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STOCHASTIC DIFFERENTIAL EQUATIONS OCCURRING IN THE ESTIMATION OF CONTINUOUS PARAMETER STOCHASTIC PROCESSES*

by

G. Kallianpur and C. Striebel

University of Minnesota

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1. Introduction. The problem of estimation which motivates the work of this paper may be described as follows. We are interested in the estimation of a "system process" $x(t)$, $0 \le t \le T$ which we assume to be defined as a stochastic process $x(t,\eta)$ on a known probability space $(\Omega_X, \mathcal{B}_X, P_X)$, $\eta \in \Omega_X$. It is further assumed that the system process cannot be observed directly. Instead we have available an "observation process" $z(\tau)$ which is given by

$$(1.1) \qquad z(\tau) = \int_0^\tau x(u)du + w(\tau) , \qquad 0 \le \tau \le T,$$

where $w(\tau)$ is a standard Wiener process independent of the system process. The available data is $z(\tau)$, $0 \le \tau \le t$, for t fixed in the interval $[0,T]$, and using this data it is required to estimate some functional of the system process

$$(1.2) \qquad G[x(\tau,\eta), \; 0 \le \tau \le T] .$$

It will be assumed that the resulting function $g(\eta)$ defined on $(\Omega_X, \mathcal{B}_X, P_X)$ by (1.2) is integrable. The process $x(t,\eta)$ is assumed to be jointly measurable and to satisfy

$$(1.3) \qquad \int_0^T x^2(t,\eta)dt < \infty \qquad \text{a.s. } P_X .$$

The system process, or more precisely, the space Ω_X on which it is defined corresponds to the parameter space in the usual Bayes approach to the theory of estimation. Thus the probability P_X is the a-priori distribution for the unknown parameter; the process $z(\tau)$, $0 \le \tau \le t$ is the observed random variable and we wish to estimate the function $g(\eta)$ on the parameter space. Hence we wish to find an estimate $\delta(z(\tau),\ 0 \le \tau \le t)$ which minimizes

$$(1.4) \qquad E(g - \delta)^2 .$$

It is well known that this is accomplished by letting

$$(1.5) \qquad \delta = E[g|z(\tau),\ 0 \le \tau \le t] .$$

We will use the shortened notation $E^t[g]$ for (1.5).

By the proper selection of the function g, (1.5) can be used to solve the smoothing, filtering and estimation problems in addition to many others. For smoothing let

$$(1.6) \qquad g(\eta) = x(s,\eta) \quad \text{where} \quad 0 \le s < t,$$

for prediction

$$(1.7) \qquad g(\eta) = x(s,\eta) \quad \text{where} \quad t < s \le T,$$

and for filtering or estimation (which is the case considered in detail in this paper),

$$(1.8) \qquad g(\eta) = x(t,\eta) .$$

A formula for the conditional expectation (1.5) where, in addition to the usual measurability and integrability assumptions, the only assumption on the system process is that it satisfy (1.3), has been derived by us in an earlier paper [9]. The Corollary to Theorem 3 of [9] which gives the form of this formula appropriate for our purpose is stated without proof as Theorem 2.1 of Section 2 and is the starting point of our investigations in this paper. The expressions for the conditional expectation given there is useful in applications only when t is fixed. If the data is coming in continuously --as in most engineering and other technological application--and we require an estimate which is being continuously revised to take into account the new data, then this formula, while valid, is not practical since the estimate at time $t+\Delta$ must be recomputed using all the past data. The value of the estimate at time t is of no help in computing the estimate at time $t+\Delta$. A practical method of computing an estimate which depends continuously on time is furnished by a stochastic differential equation. The derivation of such stochastic differential equations under the natural assumption of a Markov system process is the central problem studied in this paper. The point of departure for the derivation is Theorem 2.1. In Section 3 are given the definition and the important properties of Ito's stochastic integrals and differentials. Although this material is not new we have presented it here in some detail partly for convenience but mainly because we have been unable to find references for the precise form in which some of the results are needed by us. Section 3 also contains, along with Section 4, new lemmas which form crucial steps in the proofs of the main theorems and some of which (e.g. Lemmas 3.5 and 4.5) might have an intrinsic interest. Section 5 defines the generalized infinitesimal generator G_t and the extended operator G_t^*, along with their respective domains $\mathcal{D}$ and $\mathcal{D}^*$, associated with the Markov system process $x(t)$. Here we have worked out in detail (since no reference seems to be available) the succinct suggestions made in Dynkin's books ([3], [4]) of extending the well-known theory of the infinitestimal operator of a homogeneous (or stationary) Markov process to the general case.

The principal results of this paper, the derivation of stochastic differential equations for $E^t[f(x(t))]$ where f belongs to a suitably wide class of functions, is given in Sections 6 and 7. Theorem 6.1 is concerned with the Ito stochastic differential equation for $E^t[f(x(t))]$ while in Theorem 7.1 of Section 7 we present a rigorous derivation of a stochastic differential equation involving a new type of stochastic integral. In a purely heuristic manner it was obtained by R. L. Stratonovich in [18] and also by C. T. Striebel in [20] in which the argument (consisting of a formal passage to the limit from the discrete case) was admittedly nonrigorous and no justification was attempted. In a later paper [19] Stratonovich has drawn attention to the discrepancy between his equation and the

Ito equation and proposed a new definition of the stochastic integral which he calls a "symmetrized" integral. Unaware of his comments we announced in an abstract at The International Congress of Mathematicians (Moscow 1966) a rigorous derivation of the equation of [18] and [20] based on the properties of the stochastic integral introduced recently by D. Fisk in his thesis [5]. The idea underlying both integrals is the same but Stratonovich restricts the definition of his integral to a very special class of processes, and consequently, it cannot be used to deduce Theorem 7.1. We need the Fisk integral to do this. Section 7 is devoted to the definition and relevant properties of quasi-martingales and the integral defined for such processes. Lemma 7.1 (not found in [5]) which is used in the proof of Theorem 7.1 gives an important formula connecting the Ito integral with the Fisk-Statonovich integral where the processes concerned are quasi-martingales possessing stochastic differentials.

The two main theorems of this paper are proved under conditions that do not require the system process to be a diffusion process. Nevertheless since the diffusion processes are important examples of system processes, we treat them separately in Section 8 where the specialization of Theorems 6.1 and 7.1 to these processes is carried out.

In obtaining a general Bayes formula for the conditional expectation $E^t[f(x(t))]$ and then using it to establish the stochastic differential equation we have followed and developed the approach of Wonham who has given a rigorous proof of the Ito equation satisfied by the posterior probabilities for the very special case when $x(t)$ is a Markov process with a finite number of states, [21]. In a field where most of the work--although of a pioneering and interesting nature--has been undertaken from a purely formal point of view, Wonham's paper is written with a refreshing clarity and precision and with a recognition of the difficulties in the way of developing a general mathematical theory. The same general approach has also been outlined in a short note by Bucy [1] which, however, fails to come to grips with the difficulties inherent in the problem. Mention must be made of the formal derivations presented in the work of Kushner ([11], [12]). In his latest paper [13], (which the author has been kind enough to show us prior to its publication) he has attempted to provide a rigorous basis for his earlier results. It may not be out of place to make a few comments on the relationship of his results to ours. First of all, he does not obtain the Fisk-Stratonovich equation for $E^t[f(x(t))]$. In his derivation of the Ito stochastic differential equation the system process is assumed to be a diffusion process and as such are less general than Theorem 6.1. Secondly, his principal theorem is proved under a number of conditions whose cumulative effect on its range of validity is far from clear to us and is not explored by the author. For instance, his condition (A4) for the one dimensional case (the processes considered in [13] are vector valued) is very restrictive and excludes the simple case where $g(x(t),t) = x(t)$ and $x(t) \equiv x_0$, a standard Normal variable (see the beginning of Section 8). It would thus appear that Theorem 8.1 of Section 8 is applicable to a wider class of system diffusion processes than that of [13].

We should also like to mention a very interesting recent abstract by A. N. Shiryaev [16]. He considers two models. In one of them the system process is the (unobservable) component of a two-dimensional diffusion process

$$
\begin{aligned}
dx(t) &= a[x(t),z(t),t]dt+b_1[x(t),z(t),t]dw_1+b_2[x(t),z(t),t]dw_2 \\
dz(t) &= A[x(t),z(t),t]dt+B_1[x(t),z(t),t]dw_1+B_2[x(t),z(t),t]dw_2
\end{aligned}
\tag{1.9}
$$

where w_1 and w_2 are independent Wiener processes. The conditions under which the stochastic differential equation is obtained in this case are not fully stated.

The second model is similar to the one considered in this paper, except that $x(t)$, the system process is a Markov process with a discrete state space and the observation process $z(t)$ is related to $x(t)$ by

$$dx(t) = A[x(t),z(t),t] + B[x(t),z(t),t]dw(t). \tag{1.10}$$

If $A[x(t),z(t),t] = x(t)$ and $B \equiv 1$ we have (1.1). The conditions assumed by Shiryaev are stronger than those imposed in Theorems 6.1 and 7.1 of this paper. It can be seen that the dependence of A and B on $z(t)$ makes Shiryeav's model, in some respects, more general than ours.

Certain generalizations of the model (1.1) can easily be handled by the methods of this paper. For example, both the system and the observation processes may be vector valued and the observation equation (1.1) may be replaced by

$$z(t) = \int_0^t h[\tau,x(\tau)]d\tau + w(t) \qquad 0 \le t \le T \tag{1.11}$$

where h satisfies appropriate conditions. However, as each of these generalizations introduces complications in notation and technique and would consequently overburden the paper, it was deemed best at this stage to treat the simplest case (1.1) which includes what we consider to be the essential difficulties present in the problem.

A generalization of quite a different character, is the possibility of stochastic control in the distribution of the system process. Loosely speaking, in a "controlled" system at any given time t the distibution of the future of the system process $\{x(\tau) \text{ for } \tau > t\}$ is permitted to depend on the part of the observation process $\{z(\tau) \text{ for } 0 \le \tau \le t\}$. Shiryeav's model (1.9) is an intermediate special case of "instantaneous" control. We believe that the methods of this paper can be adapted to cover this case as well as (1.10) but more general problems involving control remain to be investigated. We hope to return to these questions in a later paper.

2. <u>A Formula for the Conditional Expectation</u>. In this section we will state without proof results from Kallianpur and Striebel [9]. Though Theorem 2.1 is of interest in its own right no discussion of its uses will be given here. The form of the theorem is modified slightly from [9] so as to make its application in later sections more convenient.

We begin by establishing some notation. Let $R^{[0,t]}$ be the space of all real-valued functions $z(\tau)$ for $0 \le \tau \le t$, let $\mathcal{B}_R^{[0,t]}$ be the product σ-field in $R^{[0,t]}$ defined in the usual manner, and let $C[0,t]$ be the space of real-valued continuous functions on the interval $[0,t]$. Define measurable spaces $(W,\mathcal{B}_W)$ and $(Z_t,\mathcal{B}_{Z_t})$ as follows

$$
(2.1) \qquad
\begin{aligned}
W &= C[0,T] \\
\mathcal{B}_W &= W \cap \mathcal{B}_R^{[0,T]} \\
Z_t &= C[0,t] \\
\mathcal{B}_{Z_t} &= Z_t \cap \mathcal{B}_R^{[0,t]}
\end{aligned}
$$

where $0 < t \le T$.

It will be assumed that a Wiener measure P_W is defined on $(W,\mathcal{B}_W)$ and that a probability space $(\Omega_X,\mathcal{B}_X,P_X)$ is also given. We will define two product spaces

$$
(2.2) \qquad (\Omega,\mathcal{A},P) = (\tilde{\Omega}_X,\tilde{\mathcal{B}}_X,\tilde{P}_X) \times (W,\mathcal{B}_W,P_W)
$$

and

$$
(2.3) \qquad
\begin{aligned}
(\hat{\Omega},\hat{\mathcal{A}},\hat{P}) &= (\Omega_X,\mathcal{B}_X,P_X) \times (\tilde{\Omega}_X,\tilde{\mathcal{B}}_X,\tilde{P}_X) \times (W,\mathcal{B}_W,P_W) \\
&= (\Omega_X,\mathcal{B}_X,P_X) \times (\Omega,\mathcal{A},P),
\end{aligned}
$$

where the spaces $(\Omega_X,\mathcal{B}_X,P_X)$ and $(\tilde{\Omega}_X,\tilde{\mathcal{B}}_X,\tilde{P}_X)$ are identical. Elements of these spaces will be denoted by

$$\eta \in \Omega_X$$

$$\tilde{\eta} \in \tilde{\Omega}_X$$

$$w \in W \tag{2.4}$$

$$w = (\tilde{\eta}, w) \in \Omega$$

$$\tilde{w} = (\eta, \tilde{\eta}, w) = (\eta, w) \in \tilde{\Omega} .$$

Define the projection transformation

$$\Phi : (\Omega, \mathcal{A}) \to (\tilde{\Omega}_X, \mathcal{B}_X) \tag{2.5}$$

by

$$\Phi(\tilde{\eta}, w) = \tilde{\eta} . \tag{2.6}$$

It will be assumed that a real-valued stochastic process $x(u,\eta)$, $0 \le u \le T$, $\eta \in \Omega_X$, called the system process, is defined on $(\Omega_X, \mathcal{B}_X, P_X)$. For $0 < t \le T$, we shall define the transformations

$$H_t : (\Omega, \mathcal{A}) \to (Z_t, \mathcal{B}_t) \tag{2.7}$$

by

$$H_t(\tilde{\eta}, w)(\tau) = h(\tilde{\eta}, \tau) + w(\tau) \qquad 0 \le \tau \le t \tag{2.8}$$

where

$$h(\tilde{\eta}, \tau) = \begin{cases} \int_0^\tau x(u,\tilde{\eta})du & \text{for } 0 \le \tau \le t \text{ if } \int_0^t [x(u,\tilde{\eta})]^2 du < \infty \\ 0 & \text{for } 0 \le \tau \le t \text{ if } \int_0^t [x(u,\tilde{\eta})]^2 du = \infty . \end{cases} \tag{2.9}$$

We quote the following lemma without proof ([9], Lemma 1).

<u>Lemma 2.1</u> <u>If the system process</u> $x(u,\eta)$ <u>is jointly measurable, then for each</u> t, $0 < t \le T$, <u>the transformation</u> H_t (2.7) <u>defined by</u> (2.8) <u>and</u> (2.9) <u>is measurable</u>.

From Lemma 2.1, the σ-field

$$\mathcal{A}_Z^t = H_t^{-1} \mathcal{B}_{Z_t} \tag{2.10}$$

is a sub σ-field on $(\Omega, \mathcal{G})$. Clearly $\mathcal{G}_Z^t$ is the σ-field generated by $z(\tau, \tilde{\eta}, w)$ for $0 \le \tau \le t$ and $z(\tau)$ given by (1.1). We are now in a position to make more precise the conditional expectations which we wish to consider in (1.5). For g an integrable random variable defined on $(\Omega_X, \mathcal{B}_X, P_X)$ we introduce the shortened notation

$$E^t[\tilde{g}](\omega) = E^{\mathcal{G}_Z^t}[g\Phi](\omega) . \tag{2.11}$$

The notation $\tilde{g}$ indicates that g is to be treated as a function of $\tilde{\eta}$ or more precisely of $(\tilde{\eta}, w)$. That is,

$$\tilde{g}(\tilde{\eta}, w) = (g\Phi)(\tilde{\eta}, w) = g(\tilde{\eta}) . \tag{2.12}$$

It is conditional expectations of the form (2.11) which are implied by (1.5). A formula for these conditional expectations is provided in the following theorem.

<u>Theorem</u> 2.1 <u>Let</u> $x(\tau, \eta)$, $0 \le \tau \le T$, $\eta \in \Omega_X$ <u>be a jointly measurable process such that</u>

$$\int_0^T [x(u,\eta)]^2 du < \infty \qquad \text{a.s. } P_X . \tag{2.13}$$

<u>Then for</u> $0 < t \le T$

$$0 < \int_{\Omega_X} \left[e^{\int_0^t x(u,\eta)x(u,\tilde{\eta})du + \int_0^t x(u,\eta)dw(u) - \frac{1}{2}\int_0^t [x(u,\eta)]^2 du} \right] P_X(d\eta) < \infty \qquad \text{a.s. } P \tag{2.14}$$

<u>and</u>

$$E^t[\tilde{g}](\tilde{\eta}, w) = \frac{\int_{\Omega_X} \left[g(\eta) e^{\int_0^t x(u,\eta)x(u,\tilde{\eta})du + \int_0^t x(u,\eta)dw(u) - \frac{1}{2}\int_0^t [x(u,\eta)]^2 du} \right] P_X(d\eta)}{\int_{\Omega_X} \left[e^{\int_0^t x(u,\eta)x(u,\tilde{\eta})du + \int_0^t x(u,\eta)dw(u) - \frac{1}{2}\int_0^t [x(u,\eta)]^2 du} \right] P_X(d\eta)} \qquad \text{a.s. } P \tag{2.15}$$

for all integrable random variables g *on* $(\Omega_X, \mathcal{B}_X, P_X)$.

This theorem follows immediately from the corollary in Section 5 of [9].

3. Stochastic Integrals and Differentials of Ito Type. In this section, we will present the properties of stochastic integrals and stochastic differential equations which will be required in later sections. Lemma 3.1 is concerned with measurability properties for processes on product spaces. Lemma 3.2 is a Fubini theorem for stochastic integrals which can be taken as integrals of sample functions. In Lemma 3.3 are collected the principal properties of stochastic integrals with respect to a Wiener process. It is due primarily to Ito [7] with some modifications as given in Gikhman and Skorokhod [6] and is stated here for convenient reference. Lemma 3.4 is concerned with convergence of stochastic integrals. Lemmas 3.5 and 3.6 give new Fubini type theorems for stochastic integrals with respect to a Wiener process. Lemma 3.6 is the principal result of this section. It is crucial in the derivation of the stochastic differential equations of Sections 4 and 6. Lemma 3.7 is also due to Ito [8] and is quoted without proof.

We shall be concerned with stochastic integrals defined with respect to the system

$$(3.1) \qquad (\Omega, \mathcal{G}, P),\ \mathcal{F}_t,\ w(t,\omega) \qquad 0 \le t \le T,\ \omega \in \Omega$$

where $(\Omega, \mathcal{G}, P)$ is an arbitrary probability space. In this section the structure (2.2) is not required. We will assume only that the $\mathcal{F}_t$ are complete with respect to P, that

$$(3.2) \qquad \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{G} \qquad \text{for } 0 \le t_1 \le t_2 \le T,$$

and that $w(t,\omega)$ is a Wiener process such that

$$(3.3) \qquad w(t,\cdot) \text{ is } \mathcal{F}_t\text{-measurable for } 0 \le t \le T,$$

and for $0 \le t \le T$

$$(3.4) \qquad \mathcal{F}_t \text{ is independent of } w(v) - w(t) \qquad \text{for } t \le v \le T.$$

The notation $\overline{\mathcal{G}}$ indicates the completion of the σ-field with respect to P. We shall consider two classes of processes defined on $(\Omega, \mathcal{G}, P)$. The class $\mathcal{M}_1$ consists of those processes $a(t,\omega)$, $0 \le t \le T$, $\omega \in \Omega$ which satisfy the following conditions:

$$(3.5) \qquad a(t,\omega) \text{ is measurable on } ([0,T] \times \Omega, \overline{\mathcal{B}_{[0,T]} \times \mathcal{G}}, \mu_{[0,T]} \times P)$$

where $\mathcal{B}_{[0,T]}$ is the class of Borel subsets of the interval $[0,T]$ and $\mu_{[0,T]}$ is Lebesgue measure on $[0,T]$,

$$(3.6) \qquad a(t,\cdot) \text{ is } \mathcal{F}_t\text{-measurable} \qquad \text{a.e. } \mu_{[0,T]},$$

and

$$(3.7) \qquad \int_0^T |a(t,\omega)|dt < \infty \qquad \text{a.s. } P .$$

The class $\mathcal{M}_2$ is defined to consist of those processes $b(t,\omega)$, $0 \le t \le T$, $\omega \in \Omega$ which satisfy (3.5), (3.6) and

$$(3.8) \qquad \int_0^T [b(t,\omega)]^2 dt < \infty \qquad \text{a.s. } P .$$

Let $(\Omega_X, \mathcal{B}_X, P_X)$ be a probability space and define

$$(3.9) \qquad (\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P}) = (\Omega_X, \mathcal{B}_X, P_X) \times (\Omega, \mathcal{G}, P)$$

$$(3.10) \qquad \tilde{\mathcal{F}}_t = \overline{\mathcal{B}_X \times \mathcal{F}_t} \qquad 0 \le t \le T .$$

It is clear then that the product system

$$(3.11) \qquad (\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P}), \ \tilde{\mathcal{F}}_t, \ w(t,\tilde{\omega}) \qquad 0 \le t \le T, \ \tilde{\omega} \in \tilde{\Omega}$$

satisfies the conditions (3.2), (3.3) and (3.4) where $w(t)$ on $\tilde{\Omega}$ is defined by

$$(3.12) \qquad w(t,\tilde{\omega}) = w(t,\eta,\omega) = w(t,\omega) .$$

The classes of processes for the product system (3.11) defined by (3.5), (3.6), (3.7) and (3.5), (3.6), (3.8) will be denoted by $\tilde{\mathcal{M}}_1$

and $\widetilde{\mathcal{M}}_2$.

Lemma 3.1 If $a \in \widetilde{\mathcal{M}}_i$,

$$(3.13)\qquad \int_0^T \left|\int_{\Omega_X} a(u,\eta,\omega)P_X(d\eta)\right|^i du < \infty \qquad \text{a.s. } P$$

and either

$$(3.14)\qquad 0 \le a(u,\eta,\omega) \qquad \text{a.e. } \mu_{[0,T]} \times \tilde{P}$$

or

$$(3.15)\qquad \int_{\Omega_X} |a(u,\eta,\omega)| P_X(d\eta) < \infty \qquad \text{a.e. } \mu_{[0,T]} \times P,$$

then

$$(3.16)\qquad \int_{\Omega_X} a(u,\eta,\omega)P_X(d\eta) \in \mathcal{M}_i$$

for $i = 1,2$.

Proof: Since $a \in \widetilde{\mathcal{M}}_i$, (3.5) holds for the system (3.11). Thus $a(t,\eta,\omega)$ is measurable with respect to $\overline{\mathcal{B}_{[0,T]} \times \mathcal{G}_X \times \mathcal{G}}$. From the Fubini theorem for non-negative functions the integrals

$$(3.17)\qquad \int_{\Omega_X} a^+(t,\eta,\omega)P_X(d\eta) \quad \text{and} \quad \int_{\Omega_X} a^-(t,\eta,\omega)P_X(d\eta)$$

are measurable with respect to $\overline{\mathcal{B}_{[0,T]} \times \mathcal{G}}$. If (3.14) holds, $a = a^+$ and hence the integral in (3.16) satisfies (3.5) for the system (3.1). If (3.15) holds then the integrals (3.17) are finite a.e. $\mu_{[0,T]} \times P$ and hence the integral in (3.16) satisfies (3.5).

From (3.6) for the product system (3.11) $a(t,\eta,\omega)$ is $\tilde{\mathcal{F}}_t$-measurable for almost all t. For t such that $a(t,\eta,\omega)$ is measurable with respect to $\tilde{\mathcal{F}}_t = \overline{\mathcal{B}_X \times \mathcal{F}_t}$, by the Fubini theorem for nonnegative functions on $\Omega_X \times \Omega$ the integrals (3.17) are $\mathcal{F}_t$-measurable. For t such that (3.14) holds a.s. $\tilde{P}$, $a^+ = a$ and the integral in (3.16) satisfies (3.6). For t such that (3.15) holds a.s. P, the integrals (3.17) are finite a.s. P and their difference

(3.16) satisfies (3.6). Thus in either case (3.6) is satisfied for almost all t.

Thus the integral (3.16) satisfies the measurability conditions (3.5) and (3.6) and it follows from assumption (3.13) that (3.16) is satisfied.

Lemma 3.2 If $a \in \tilde{\mathcal{M}}_1$ and

(3.18) $$\int_0^T [\int_{\Omega_X} |a(t,\eta,\omega)| P_X(d\eta)]dt < \infty \qquad \text{a.s. } P$$

then

(3.19) $$\int_{\Omega_X} a(t,\eta,\omega) P_X(d\eta) \in \mathcal{M}_1 ,$$

(3.20) $$\int_{\Omega_X}\int_0^T a(t,\eta,\omega) dt P_X(d\eta) = \int_0^T \int_{\Omega_X} a(t,\eta,\omega) P_X(d\eta) dt \qquad \text{a.s. } P$$

and the integrals (3.20) are finite a.s. P.

Proof: Assumption (3.18) implies both (3.13) and (3.15) (with $i = 1$), so that (3.19) follows from (3.16) of Lemma 1.

Since $a(t,\eta,\omega)$ is measurable with respect to $\overline{\mathcal{B}_{[0,T]} \times \mathcal{B}_X \times \mathcal{G}}$, for almost all ω $a(t,\eta,\omega)$ is measurable with respect to $\overline{\mathcal{B}_{[0,T]} \times \mathcal{B}_X}$. For ω such that (3.18) holds and $a(t,\eta,\omega)$ is $\overline{\mathcal{B}_{[0,T]} \times \mathcal{B}_X}$ measurable, the Fubini theorem for absolutely integrable functions on $[0,T] \times \Omega_X$ implies (3.20) and the finiteness of the integrals.

Lemma 3.3 For $b \in \mathcal{M}_2$, the stochastic integral

(3.21) $$\int_0^T b(t,\omega) dw(t,\omega)$$

is a finite random variable on $(\Omega, \overline{\mathcal{G}}, P)$. It is determined uniquely a.s. P by the following properties:

i) if $b \in \mathcal{M}_2$ is a step function

(3.22) $$b(t,\omega) = b_i(\omega) \qquad \text{for } t_i \le t < t_{i+1}$$

where

(3.23) $$0 = t_0 < t_1 < \ldots < t_m = T,$$

then

(3.24) $$\int_0^T b(t,\omega)dw(t,\omega) = \sum_{i=0}^{m-1} b_i(\omega)(w(t_{i+1},\omega) - w(t_i,\omega)) \quad \text{a.s. } P$$

and

ii) if $b_n \in \mathcal{M}_2 \quad n = 0,1,2,\ldots$

and

(3.25) $$\int_0^T [b_n(t)-b_0(t)]^2 dt \overset{P}{\to} 0 ,$$

then

(3.26) $$\int_0^T b_n(t)dw(t) \overset{P}{\to} \int_0^T b_0(t)dw(t).$$

The stochastic integral also satisfies:

iii) for $b_1, b_2 \in \mathcal{M}_2$ and β_1, β_2 real numbers

(3.27) $$\int_0^T [\beta_1 b_1(t)+\beta_2 b_2(t)]dw(t) = \beta_1\int_0^T b_1(t)dw(t)+\beta_2\int_0^T b_2(t)dw(t) \quad \text{a.s. } P$$

and

iv) if $b \in \mathcal{M}_2$ and $\int_s^t E^{\mathcal{F}_s}[b^2(u)]du < \infty$ a.s. for

$0 \leq s \leq u \leq t \leq T$, then

(3.28) $$E^{\mathcal{F}_s}[\int_s^t b(u)dw(u)]^2 = \int_s^t E^{\mathcal{F}_s}[b^2(u)]du \quad \text{a.s. } P .$$

The notation in (3.25) and (3.26) indicates convergence in probability P. This lemma is due to Ito [7] (Theorem 7.1 p. 14) except for the properties (ii) and (iv) which are found in [6] pp. 492-493. It clearly holds when the system (3.1) is replaced by the product system (3.11). Integrals on the restricted range [s,t] as in (3.28) are obtained by considering functions $b \in \mathcal{M}_2$ for which

(3.29) $$b(u,\omega) = 0 \qquad \text{for } 0 \leq u < s \text{ and } t < u \leq T .$$

Lemma 3.4 For each t in $[0,T]$ let

$$a_n(t) \to a_0(t) \qquad \text{a.s. } P \tag{3.30}$$

and for $n = 1,2,\ldots$

$$|a_n(t)| \le \bar{a}(t) \qquad \text{a.s. } P\ . \tag{3.31}$$

(i) If

$$\bar{a} \in \mathcal{M}_1 \quad \text{and} \quad a_n \in \mathcal{M}_1 \quad \text{for} \quad n = 0,1,2,\ldots \tag{3.32}$$

then

$$\int_0^T a_n(t)dt \xrightarrow{P} \int_0^T a_0(t)dt\ . \tag{3.33}$$

(ii) If

$$\bar{a} \in \mathcal{M}_2 \quad \text{and} \quad a_n \in \mathcal{M}_2 \quad \text{for} \quad n = 0,1,2,\ldots, \tag{3.34}$$

then

$$\int_0^T a_n(t)dw(t) \xrightarrow{P} \int_0^T a_0(t)dw(t)\ . \tag{3.35}$$

Proof: On $[0,T] \times \Omega$, let

$$A = \{(t,\omega) | a_n(t,\omega) \to a_0(t,\omega) \text{ and } |a_n(t,\omega)| \le \bar{a}(t,\omega),\ n=1,2,\ldots\}. \tag{3.36}$$

From (3.30) and (3.31), for $0 \le t \le T$

$$P(A_t) = 1 \tag{3.37}$$

where A_t is the section of A at t. Thus by the Fubini Theorem

$$(\mu_{[0,T]} \times P)(A) = \int_0^T P(A_t)dt = T\ . \tag{3.38}$$

It follows that

$$a_n(t,\omega) \to a_0(t,\omega) \qquad \text{a.e. } \mu_{[0,T]} \times P \tag{3.39}$$

and

$$|a_n(t,\omega)| \le \bar{a}(t,\omega) \qquad \text{a.e. } \mu_{[0,T]} \times P\ . \tag{3.40}$$

For $i = 1,2$ and $N = 1,2,\ldots$ let

(3.41) $$B_{N,i} = \{\omega | 2^{i-1}\int_0^T |a_0(t,\omega)|^i dt + 2^{i-1}\int_0^T |\bar{a}(t,\omega)|^i dt < N\}.$$

From (3.32) for $n = 0$

(3.42) $$P(B^c_{N,1}) \to 0 \qquad \text{as} \quad N \to \infty .$$

From (3.34) for $n = 0$

(3.43) $$P(B^c_{N,2}) \to 0 \qquad \text{as} \quad N \to \infty .$$

For $\epsilon > 0$, choose N such that

(3.44) $$P(B^c_N) \leq \frac{\epsilon}{2} .$$

Then from (3.44) and the Chebychev inequality

(3.45) $$P[\int_0^T |a_n(t)-a_0(t)|^i dt > \epsilon] \leq P(B^c_{N,i})+P[B_{N,i} \cap \{\int_0^T |a_n(t)-a_0(t)|^i dt > \epsilon\}]$$

$$\leq \frac{\epsilon}{2} + \frac{1}{\epsilon} E[I_{B_{N,i}} \int_0^T |a_n(t)-a_0(t)|^i dt] .$$

From elementary inequalities and (3.40)

(3.46) $$I_{B_{N,i}}(\omega)|a_n(t,\omega)-a_0(t,\omega)|^i \leq I_{B_{N,i}}(\omega).2^{i-1}[|\bar{a}(t,\omega)|^i+|a_0(t,\omega)|^i]$$

$$\text{a.e. } \mu_{[0,T]} \times P .$$

From (3.39) the left side of (3.46) goes to zero a.e. $\mu_{[0,T]} \times P$. The right of (3.46) is integrable with respect to $\mu_{[0,T]} \times P$, since from (3.41) and the Fubini Theorem for non-negative functions

(3.47) $$\int[\int_0^T I_{B_{N,i}}(\omega)\, 2^{i-1}[|\bar{a}(t,\omega)|^i+|a_0(t,\omega)|^i]dt]P(d\omega) \leq N .$$

Thus by the dominated convergence theorem

(3.48) $$E[I_{B_{N,i}} \int_0^T |a_n(t)-a_0(t)|^i dt] \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty .$$

It follows from (3.45) that

$$\int_0^T |a_n(t)-a_0(t)|^i dt \xrightarrow{P} 0 \qquad \text{as} \quad n \to \infty . \tag{3.49}$$

For $i = 1$, the result (3.33) follows from (3.49) and the inequality

$$\left|\int_0^T a_n(t)dt - \int_0^T a_0(t)dt\right| \le \int_0^T |a_n(t) - a_0(t)|dt . \tag{3.50}$$

For $i = 2$, the result (3.35) follows from (3.34), (3.49) and (ii) of Lemma 3.3.

<u>Lemma</u> 3.5 <u>If</u> $b \in \widetilde{\mathcal{M}}_2$ <u>and</u>

$$\int_s^t \tilde{E}[b(u)]^2 du < \infty \tag{3.51}$$

<u>then</u>

$$\int_s^t \left[\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)\right]dw(u,\omega) = \int_{\Omega_X}\left[\int_s^t b(u,\eta,\omega)dw(u,\omega)\right]P_X(d\eta) \quad \text{a.s. } P \tag{3.52}$$

<u>and the integrals</u> (3.52) <u>are finite</u> a.s. P.

Proof: Since only integrals on the range $s \le u \le t$ are involved in (3.52), we shall assume that $b(u)$ is zero outside this range.

By the Schwartz inequality, the Fubini Theorem for non-negative functions, (3.9), and assumption (3.51)

$$\begin{aligned}\int_\Omega \int_0^T \left[\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)\right]^2 du P(d\omega) &\le \int_\Omega \int_0^T \int_{\Omega_X} [b(u,\eta,\omega)]^2 P_X(d\eta) du P(d\omega) \\ &= \int_0^T \int_\Omega \int_{\Omega_X} [b(u,\eta,\omega)]^2 P_X(d\eta)P(d\omega)du = \int_0^T \tilde{E}[b(u)]^2 du < \infty .\end{aligned} \tag{3.53}$$

Thus (3.13) of Lemma 3.1 is satisfied for b and $i = 2$. Similarly

$$\begin{aligned}&\int_0^T \int_\Omega \left[\int_{\Omega_X} |b(u,\eta,\omega)|P_X(d\eta)\right]^2 P(d\omega)du \\ &\le \int_0^T \int_\Omega \int_{\Omega_X} [b(u,\eta,\omega)]^2 P_X(d\eta)P(d\omega)du = \int_0^T \tilde{E}[b(u)]^2 du < \infty .\end{aligned} \tag{3.54}$$

From (3.54), it is seen that (3.15) of Lemma 3.1 is satisfied and we conclude that

$$\int_{\Omega_X} b(u,\eta,\omega) P_X(d\eta) \in \mathfrak{M}_2 \tag{3.55}$$

and hence from Lemma 3.3 the integral

$$\int_s^t \Big[\int_{\Omega_X} b(u,\eta,\omega) P_X(d\eta)\Big] dw(u) \tag{3.56}$$

is well defined. Again from the Schwartz inequality, (3.9), (iv) of Lemma 3.3 and assumption (3.51)

$$\begin{aligned}
&\int_\Omega \int_{\Omega_X} \Big|\int_s^t b(u,\eta,\omega) dw(u,\omega)\Big| P_X(d\eta) P(d\omega) \leq \sqrt{\tilde{E}\Big[\int_s^t b(u) dw(u)\Big]^2} \\
&= \sqrt{\tilde{E}\,\tilde{E}^{\tilde{\mathfrak{F}}_s}\Big[\int_s^t b(u) dw(u)\Big]^2} = \sqrt{\int_s^t \tilde{E}[b(u)]^2 du} < \infty .
\end{aligned} \tag{3.57}$$

Thus

$$\int_s^t b(u,\eta,\omega) dw(u,\omega) \tag{3.58}$$

is integrable on $\Omega_X \times \Omega$. It follows then from the Fubini Theorem that

$$\int_{\Omega_X} \Big[\int_s^t b(u,\eta,\omega) dw(u,\omega)\Big] P_X(d\eta) \tag{3.59}$$

is finite a.s. P and $\overline{\mathfrak{G}}$ - measurable.

In the discussion of stochastic integrals in Doob ([2] Chapter IX, Section 5) the integral is defined under the assumption that $b \in \tilde{\mathfrak{M}}_2$ and satisfies (3.51). Under these assumptions it is shown that there exist step functions $b_n \in \tilde{\mathfrak{M}}_2$ given by

$$b_n(u,\eta,\omega) = b_i^n(\eta,\omega) \qquad t_i^n \leq u < t_{i+1}^n \tag{3.60}$$

where

$$s = t_0^n < t_1^n < \ldots < t_{m_n}^n = t \tag{3.61}$$

and such that

(3.62) $$\tilde{E}[b_i^n]^2 < \infty$$

and

(3.63) $$\int_s^t \tilde{E}[b_n(u) - b(u)]^2 du \to 0 \qquad \text{as} \quad n \to \infty$$

(see Doob [2] p. 436-441). From the linearity of the integral $P_X(d\eta)$, the linearity of the stochastic integral (iii) of Lemma 3.3, the Schwartz inequality, (iv) of Lemma 3.3 and (3.63)

$$\int_\Omega \{\int_{\Omega_X} [\int_s^t b_n(u,\eta,\omega)dw(u,\omega)]P_X(d\eta) - \int_{\Omega_X} [\int_s^t b(u,\eta,\omega)dw(u,\omega)]P_X(d\eta)\}^2 P(d\omega)$$

$$\leq \int_\Omega \{ \int_{\Omega_X} [\int_s^t (b_n(u,\eta,\omega)-b(u,\eta,\omega)) dw(u,\omega)]P_X(d\eta)\}^2 P(d\omega)$$

(3.64)
$$\leq \int_\Omega \int_{\Omega_X} [\int_s^t (b_n(u,\eta,\omega)-b(u,\eta,\omega))dw(u,\omega)]^2 P_X(d\eta)P(d\omega)$$

$$= \int_s^t \tilde{E}[b_n(u)-b(u)]^2 du \to 0 \qquad \text{as} \quad n \to \infty .$$

Thus from (3.64)

(3.65) $$\int_{\Omega_X} [\int_s^t b_n(u,\eta,\omega)dw(u,\omega)]P_X(d\eta) \to \int_{\Omega_X} [\int_s^t b(u,\eta,\omega)dw(u,\omega)]P_X(d\eta)$$

in L_2 norm on $(\Omega,\mathcal{B},P)$, and hence convergence of (3.65) holds in probability P. Since b_n is a step function given by (3.60), from (i) of Lemma 3.3

$$\int_{\Omega_X} [\int_s^t b_n(u,\eta,\omega)dw(u,\omega)]P_X(d\eta)$$

(3.66)
$$= \int_{\Omega_X} [\sum_i b_i^n(\eta,\omega)(w(t_{i+1}^n,\omega) - w(t_i^n,\omega))]P_X(d\eta)$$

$$= \sum_i [\int_{\Omega_X} b_i^n(\eta,\omega)P_X(d\eta)][w(t_{i+1}^n,\omega) - w(t_i^n,\omega)] .$$

Since the $b_n \in \tilde{\mathcal{M}}_2$, and from (3.62) they satisfy (3.51) by the argument (3.53) through (3.55), it is clear that

$$(3.67) \qquad \int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta) \in \mathcal{M}_2 .$$

Also from (3.60)

$$(3.68) \qquad \int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta) = \int_{\Omega_X} b_i^n(\eta,\omega)P_X(d\eta) \quad \text{for } t_i^n \le u \le t_{i+1}^n$$

is a step function belonging to $\mathcal{M}_2$. Thus from (i) of Lemma 3.3

$$(3.69) \qquad \int_s^t [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta)]dw(u,\omega) = \sum_i [\int_{\Omega_X} b_i^n(\eta,\omega)P_X(d\eta)][w(t_{i+1}^n,\omega)-w(t_i^n,\omega)]$$

and from (3.66)

$$(3.70) \qquad \int_{\Omega_X} [\int_s^t b_n(u,\eta,\omega)dw(u,\omega)]P_X(d\eta) = \int_s^t [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta)]dw(u,\omega) .$$

By iii) and iv) of Lemma 3.3, the Schwartz inequality and (3.63)

$$\int_\Omega \left[\int_s^t [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta)]dw(u,\omega) - \int_s^t [\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)]dw(u,\omega)\right]^2 P(d\omega)$$

$$= \int_\Omega \left[\int_s^t [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta) - \int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)]dw(u,\omega)\right]^2 P(d\omega)$$

$$(3.71) \qquad = \int_s^t \int_\Omega [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta) - \int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)]^2 P(d\omega)du$$

$$\le \int_s^t \int_\Omega \int_{\Omega_X} [b_n(u,\eta,\omega) - b(u,\eta,\omega)]^2 P_X(d\eta)P(d\omega)du$$

$$= \int_s^t \tilde{E}[b_n(u) - b(u)]^2 du \to 0 \qquad \text{as } n \to \infty .$$

Thus from (3.71)

$$(3.72) \qquad \int_s^t [\int_{\Omega_X} b_n(u,\eta,\omega)P_X(d\eta)]dw(u,\omega) \to \int_s^t [\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)]dw(u,\eta)$$

in L_2 norm on $(\Omega,\mathcal{B},P)$ and hence in probability (P). The result

(3.52) then follows from (3.65), (3.70) and (3.72), and the Lemma is proved.

Lemma 3.6 Let $b \in \tilde{\mathcal{M}}_2$ satisfy

$$(3.73) \qquad \int_s^t \Big[\int_{\Omega_X} |b(u,\eta,\omega)| P_X(d\eta)\Big]^2 du < \infty \qquad \text{a.s. } P$$

and

$$(3.74) \qquad \int_{\Omega_X} \Big\{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b^2(u)](\eta,\omega)du\Big\}^{\frac{1}{2}} P_X(d\eta) < \infty \qquad \text{a.s. } P .$$

Then

$$(3.75) \qquad \int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta) \in \mathcal{M}_2 ,$$

$$(3.76) \qquad \int_s^t \Big[\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)\Big]dw(u,\omega) = \int_{\Omega_X}\Big[\int_s^t b(u,\eta,\omega)dw(u,\omega)\Big]P_X(d\eta) \qquad \text{a.s. } P,$$

and the integrals in (3.76) are finite a.s. P.

Proof: It is easily seen from the assumption $b \in \tilde{\mathcal{M}}_2$, that

$$(3.77) \qquad |b| \in \tilde{\mathcal{M}}_2 .$$

From assumption (3.73)

$$(3.78) \qquad \int_0^T \Big[\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta)\Big]^2 du \le \int_0^T \Big[\int_{\Omega_X} |b(u,\eta,\omega)| P_X(d\eta)\Big]^2 du < \infty \qquad \text{a.s. } P,$$

Thus (3.13) of Lemma 3.1 for $i = 2$ is satisfied by both b and $|b|$.

For non-negative random variables $B \ge 0$ on $\tilde{\Omega}$, from the Fubini Theorem on $(\Omega_X \times \Omega, \mathcal{B}_X \times \mathcal{F}_s)$ and the properties of conditional expectations, it can easily be verified that

$$(3.79) \qquad E^{\mathcal{F}_s}\Big[\int_{\Omega_X} B(\eta,\omega)P_X(d\eta)\Big] = \int_{\Omega_X} \tilde{E}^{\tilde{\mathcal{F}}_s}[B](\eta,\omega)P_X(d\eta) \qquad \text{a.s. } P .$$

From this result (3.79), the Schwartz inequality for conditional expectations, the Fubini Theorem for non-negative functions, the Schwartz inequality and (3.74)

$$\int_s^t E^{\mathcal{F}_s}[\int_{\Omega_X} |b(u,\eta,\omega)| P_X(d\eta)](\omega)du$$

$$= \int_s^t [\int_{\Omega_X} \tilde{E}^{\tilde{\hat{\mathcal{F}}}_s}[|b(u)|](\eta,\omega)P_X(d\eta)]du \qquad \text{a.s. } P$$

(3.80) $$\le \int_s^t \int_{\Omega_X} \{\tilde{E}^{\tilde{\hat{\mathcal{F}}}_s}[b(u)^2](\eta,\omega)\}^{\frac{1}{2}} P_X(d\eta)du \qquad \text{a.s. } P$$

$$= \int_{\Omega_X} \int_s^t \{\tilde{E}^{\tilde{\hat{\mathcal{F}}}_s}[b(u)^2](\eta,\omega)\}^{\frac{1}{2}} du P_X(d\eta) \qquad \text{a.s. } P$$

$$\le (t-s)^{\frac{1}{2}} \int_{\Omega_X} \{\int_s^t \tilde{E}^{\tilde{\hat{\mathcal{F}}}_s}[b(u)^2](\eta,\omega)du\}^{\frac{1}{2}} P_X(d\eta) < \infty \qquad \text{a.s. } P .$$

From (3.80) it follows that

(3.81) $$\int_{\Omega_X} |b(u,\eta,\omega)| P_X(d\eta) < \infty \qquad \text{a.e. } \mu_{[0,T]} \times P$$

and hence that (3.15) of Lemma 3.1 is satisfied for both b and $|b|$. Thus from Lemma 3.1

(3.82) $$\int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta) \in \mathcal{M}_2$$

and

(3.83) $$\int_{\Omega_X} |b(u,\eta,\omega)| P_X(d\eta) \in \mathcal{M}_2 .$$

From Lemma 3.3 and (3.82) the stochastic integrals in (3.76) are well defined. Next it will be shown that

(3.84) $$\int_{\Omega_X} |\int_s^t b(u,\eta,\omega)dw(u,\omega)| P_X(d\eta) < \infty \qquad \text{a.s. } P$$

and hence that the integral on the left side of (3.76) is finite a.s. P.

From (3.79), assumption (3.74) and arguments which are by now familiar,

$$E^{\mathcal{F}_s}\Big[\int_{\Omega_X} \Big|\int_s^t b(u,\eta,\omega)dw(u,\omega)\Big| P_X(d\eta)\Big](\omega)$$

$$= \int_{\Omega_X} \tilde{E}^{\tilde{\mathcal{F}}_s}\Big[\Big|\int_s^t b(u)dw(u)\Big|\Big](\eta,\omega)P_X(d\eta) \qquad \text{a.s. } P$$

$$(3.85) \qquad \le \int_{\Omega_X} \{\tilde{E}^{\tilde{\mathcal{F}}_s}[\int_s^t b(u)dw(u)]^2(\eta,\omega)\}^{\frac{1}{2}}P_X(d\eta) \qquad \text{a.s. } P$$

$$= \int_{\Omega_X} \{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b(u)^2](\eta,\omega)du\}^{\frac{1}{2}}P_X(d\eta) < \infty \qquad \text{a.s. } P .$$

The integrability of the stochastic integral in (3.84) follows from (3.85). For $N = 1,2,\ldots$ let

$$(3.86) \qquad b_N(u,\eta,\omega) = \begin{cases} b(u,\eta,\omega) & \text{if } |b(u,\eta,\omega)| \le N \\ 0 & \text{otherwise} . \end{cases}$$

The measurability properties (3.5) and (3.6) of b are clearly preserved by the b_N processes. Since the b_N are bounded, they satisfy the requirements (3.8) and (3.51) for finite integrals. Thus $b_N \in \tilde{\mathfrak{M}}_2$ and the conclusion (3.52) of Lemma 3.5 holds for the b_N.

From the definition (3.86), since $b(u,\eta,\omega)$ is finite for all (u,η,ω),

$$(3.87) \qquad b_N(u,\eta,\omega) \to b(u,\eta,\omega)$$

and

$$(3.88) \qquad |b_N(u,\eta,\omega)| \le |b(u,\eta,\omega)| .$$

For (u,ω) such that (3.81) holds, the Lebesgue dominated convergence theorem applies and

$$(3.89) \qquad \int_{\Omega_X} b_N(u,\eta,\omega)P_X(d\eta) \to \int_{\Omega_X} b(u,\eta,\omega)P_X(d\eta) \qquad \text{a.e. } \mu_{[0,T]} \times P .$$

From (3.88)

$$(3.90) \qquad \Big|\int_{\Omega_X} b_N(u,\eta,\omega)P_X(d\eta)\Big| \le \int_{\Omega_X} |b_N(u,\eta,\omega)|P_X(d\eta) \le \int_{\Omega_X} |b(u,\eta,\omega)|P_X(d\eta) .$$

From (3.83), (ii) of Lemma 3.4 applies to the sequence in (3.89) and thus

$$\int_s^t [\int_{\Omega_X} b_N(u,\eta,\omega) P_X(d\eta)] dw(u,\omega) \xrightarrow{P} \int_s^t [\int_{\Omega_X} b(u,\eta,\omega) P_X(d\eta)] dw(u,\omega) \; . \tag{3.91}$$

From (3.87) and (3.88) where by assumption $b \in \tilde{\mathcal{M}}_2$, (ii) of Lemma 3.4 applies and

$$\int_s^t b_N(u,\eta,\omega) dw(u,\omega) \xrightarrow{\tilde{P}} \int_s^t b(u,\eta,\omega) dw(u,\omega) \; . \tag{3.92}$$

Next we shall show that

$$\int_{\Omega_X} [\int_s^t b_N(u,\eta,\omega) dw(u,\omega)] P_X(d\eta) \xrightarrow{P} \int_{\Omega_X} [\int_s^t b(u,\eta,\omega) dw(u,\omega)] P_X(d\eta) \; . \tag{3.93}$$

The result (3.76) then will follow from (3.52) for the b_N, (3.91) and (3.93). From the linearity of the stochastic integral on $\tilde{\mathcal{M}}_2$ given by (iii) of Lemma 3.3 and proceeding as in (3.85) we have

$$\begin{aligned}
& E^{\mathcal{F}_s}\{|\int_{\Omega_X} [\int_s^t b_N(u,\eta,\omega) dw(u,\omega)] P_X(d\eta) - \int_{\Omega_X} [\int_s^t b(u,\eta,\omega) dw(u,\eta)] P_X(d\eta)|\} \\
&= E^{\mathcal{F}_s}\{|\int_{\Omega_X} [\int_s^t [b_N(u,\eta,\omega) - b(u,\eta,\omega)] dw(u,\eta)] P_X(d\eta)|\} \quad \text{a.s. } P \\
&\le E^{\mathcal{F}_s}\{\int_{\Omega_X} |\int_s^t [b_N(u,\eta,\omega) - b(u,\eta,\omega)] dw(u,\eta)| P_X(d\eta)\} \quad \text{a.s. } P \\
&\le (t-s)^{\frac{1}{2}} \int_{\Omega_X} \{\int_s^t E^{\mathcal{F}_s} [b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 du\}^{\frac{1}{2}} P_X(d\eta) \quad \text{a.s. } P \; .
\end{aligned} \tag{3.94}$$

From (3.87) and (3.88)

$$[b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 \to 0 \tag{3.95}$$

and

$$[b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 \le 4[b(u,\eta,\omega)]^2 \; . \tag{3.96}$$

From assumption (3.74) it is clear that

$$(3.97) \qquad \tilde{E}^{\tilde{\mathcal{F}}_s}[b(u)^2](\eta,\omega) < \infty \qquad \text{a.e. } \mu_{[0,T]} \times \tilde{P} .$$

Thus from (3.95), (3.96) and (3.97) applying the domination convergence theorem for conditional probabilities ([14], p. 348)

$$(3.98) \qquad \tilde{E}^{\tilde{\mathcal{F}}_s}[(b_N(u,\eta,\omega)-b(u,\eta,\omega))^2] \to 0 \qquad \text{a.e. } \mu_{[0,T]} \times \tilde{P} .$$

From (3.96)

$$(3.99) \qquad \tilde{E}^{\tilde{\mathcal{F}}_s}[b_N(u,\eta,\omega)-b(u,\eta,\omega)]^2 \le 4\tilde{E}^{\tilde{\mathcal{F}}_s}[b(u,\eta,\omega)]^2 \qquad \text{a.s. } \tilde{P}$$

for all u. Again from (3.74)

$$(3.100) \qquad \int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b(u)^2](\eta,\omega)du < \infty \qquad \text{a.s. } \tilde{P}.$$

Thus from (3.98), (3.99) and (3.100) the dominated convergence theorem applies for (η,ω) fixed and such that (3.99) holds. Thus

$$(3.101) \qquad \int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 du \to 0 \qquad \text{a.s. } \tilde{P} .$$

From (3.101) and (3.99)

$$(3.102) \qquad \{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 du\}^{\frac{1}{2}} \to 0 \qquad \text{a.s. } \tilde{P}$$

and

$$(3.103) \qquad \{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b_N(u,\eta,\omega) - b(u,\eta,\omega)]^2 du\}^{\frac{1}{2}} \le 2\{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b(u,\eta,\omega)]^2 du\}^{\frac{1}{2}} \qquad \text{a.s. } \tilde{P}$$

where from (3.74)

$$(3.104) \qquad \int_{\Omega_X} \{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}[b(u)^2](\eta,\omega)du\}^{\frac{1}{2}} P_X(d\eta) < \infty \qquad \text{a.s. } P .$$

Thus for ω fixed and such that (3.103) hold a.s. P_X and (3.104) holds, by the dominated convergence theorem

$$(3.105) \qquad \int_{\Omega_X} \{\int_s^t E^{\tilde{\mathcal{F}}_s}[b_N(u,\eta,\omega,) - b(u,\eta,\omega,)]^2 du\}^{\frac{1}{2}} P_X(d\eta) \to 0 \qquad \text{a.s. } P .$$

This result with (3.94) implies that the convergence in (3.93) holds in conditional L_1 norm and hence that it holds in probability. This completes the proof of Lemma 3.6.

Following Ito ([8], p. 187) we shall say that a process $\zeta(t)$ defined on $(\Omega, \mathcal{G}, P)$ [or $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{P})$] has a stochastic differential of Ito type

$$(3.106) \qquad d\zeta(t) = a(t)dt + b(t)dw(t) \qquad 0 \le t \le T$$

provided

$$(3.107) \qquad a \in \mathcal{M}_1 \qquad (\text{or } \tilde{\mathcal{M}}_1),$$

$$(3.108) \qquad b \in \mathcal{M}_2 \qquad (\text{or } \tilde{\mathcal{M}}_2),$$

and

$$(3.109) \qquad \zeta(t) - \zeta(s) = \int_s^t a(u)du + \int_s^t b(u)dw(u) \qquad \text{a.s. } P \quad (\text{or } \tilde{P})$$

for all $0 \le s < t \le T$.

The following lemma, due to Ito, is taken from [8] (p. 187) and will be used extensively in the next section.

Lemma 3.7 *Assume that the processes* $\zeta_i(t)$, $i = 1,2,\ldots,n$ *have differentials*

$$(3.110) \qquad d\zeta_i(t) = a_i(t)dt + b_i(t)dw(t) \qquad 0 \le t \le T, \quad i = 1,2,\ldots,n.$$

Let $\Gamma(x)$ *be a real-valued function of the* n-*vector* x *such that*

$$(3.111) \qquad \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x)$$

is continuous on R^n *for* $i,j = 1,2,\ldots,n$. *Define the process*

$$(3.112) \qquad \xi(t) = \Gamma(\zeta(t)), \quad \text{where } \zeta(t) = [\zeta_1(t),\ldots,\zeta_n(t)].$$

Then $\xi(t)$ *has a differential*

$$(3.113) \qquad d\xi(t) = A(t)dt + B(t)dw(t) \qquad 0 \le t \le T$$

where

$$(3.114) \qquad A(t) = \sum_i \frac{\partial \Gamma}{\partial x_i} (\zeta(t))a_i(t) + \frac{1}{2} \sum_i \sum_j \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(\zeta(t))b_i(t)b_j(t)$$

and

$$(3.115) \qquad B(t) = \sum_i \frac{\partial}{\partial x_i} \Gamma(\zeta(t))b_i(t)$$

for $0 \le t \le T$.

It will be convenient to introduce a slightly nore general form of the Ito differential. Let $M(t)$ be a process with a stochastic differential of the usual Ito type (3.106) - (3.109)

$$(3.116) \qquad dM(t) = a_1(t)dt + b_1(t)dw(t) \qquad 0 \le t \le T$$

and let $N(t)$ be a process for which

$$(3.117) \qquad N(t)a_1(t) \in \mathcal{M}_1 \quad \text{and} \quad N(t)b_1(t) \in \mathcal{M}_2 ,$$

then we define the Ito integral by

$$(3.118) \qquad \int_0^T N(t)dM(t) = \int_0^T N(t)a_1(t)dt + \int_0^T N(t)b_1(t)dw(t)$$

and the corresponding Ito differential by

$$(3.119) \qquad N(t)dM(t) = N(t)a_1(t)dt + N(t)b_2(t)dw(t) \qquad 0 \le t \le T .$$

The following lemma can easily be proved following the argument of Skorokhod [17], p. 25, and is given to show that the more usual interpretation of the integral as a limit of Ito type sums holds under slightly stronger conditions for the integral defined by (3.116) - (3.118).

Lemma 3.8 *Let* $M(t)$ *have a differential* (3.116) *and let* $N(t)$ *also have a differential*

$$(3.120) \qquad dN(t) = a_2(t)dt + b_2(t)dw(t) \qquad 0 \le t \le T.$$

Then the Ito integral defined by (3.118) satisfies

$$\int_0^T N(t)dM(t) = \underset{n\to\infty}{p\ \lim} \sum_{\Pi_n} N(t_j^n)[M(t_{j+1}^n) - M(t_j^n)] \tag{3.121}$$

where $\max_j(t_{j+1}^n - t_j^n) \to 0$ as $n \to \infty$ for the subdivision $\Pi_n = \{t_j^n\}$ of the interval $[0,T]$.

It should be noticed that (3.117) holds since from (3.120) $N(t)$ satisfies the measurability requirements (3.5) and (3.6) and may be taken to have continuous sample functions a.s. (see, Skorokhod [17], Theorem 3, p. 21) and from (3.116) $a_1 \in \mathcal{M}_1$ and $b_2 \in \mathcal{M}_2$. Thus (3.117) is satisfied and the integral in (3.121) is well defined by (3.118).

4. Preliminary Lemmas. We consider now the problem introduced in Sections 1 and 2. The space $(\Omega, \mathcal{B}, P)$ is given by (2.2), $x(t,\eta)$ is a jointly measurable process on $(\Omega_X, \mathcal{B}_X, P_X)$ and $w(t,w)$ is a Wiener process defined on $(W, \mathcal{B}_W, P_W)$ and hence on Ω. Since the second argument in $w(t,w)$ is somewhat redundant it will be omitted as in Section 2. We shall let $\mathcal{B}_t$ be the σ-field in $\mathcal{B}_W$ induced by the process $w(u)$, $0 \le u \le t$, and define

$$(4.1) \qquad \mathcal{F}_s = \overline{\tilde{\mathcal{B}}_X \times \mathcal{B}_s} \, .$$

From (3.10) then

$$(4.2) \qquad \tilde{\mathcal{F}}_s = \overline{\mathcal{B}_X \times \mathcal{F}_s} = \overline{\mathcal{B}_X \times \tilde{\mathcal{B}}_X \times \mathcal{B}_s} \, .$$

Lemma 4.1 If the jointly measurable process $x(t,\eta)$ satisfies

$$(4.3) \qquad \int_0^T [x(t,\eta)]^2 dt < \infty \qquad \text{a.s. } P_X$$

and a process $\zeta(t)$ on $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$ satisfies

$$(4.4) \qquad \begin{aligned} \zeta(t,\tilde{\omega}) &\equiv \zeta(t,\eta,\tilde{\eta},w) \\ &= \int_0^t x(u,\eta)dw(u) + \int_0^t x(u,\eta)x(u,\tilde{\eta})du - \tfrac{1}{2}\int_0^t [x(u,\eta)]^2 du \qquad \text{a.s. } \tilde{P}, \end{aligned}$$

for all $0 \le t \le T$, then $\zeta_1(t)$ defined by

$$(4.5) \qquad \zeta_1(t,\tilde{\omega}) = e^{\zeta(t,\tilde{\omega})}$$

has a differential

$$(4.6) \qquad d\zeta_1(t) = e^{\zeta(t)}x(t)\tilde{x}(t)dt + e^{\zeta(t)}x(t)dw(t) \qquad 0 \le t \le T \, .$$

Proof: From (4.3) and its measurability on $[0,T] \times \Omega_X$, the process $x(t)$ defined on $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$ by $x(t,\eta,\tilde{\eta},w) = x(t,\eta)$ belongs to $\tilde{\mathcal{M}}_2$. The process $x(u,\eta)x(u,\tilde{\eta}) - \frac{1}{2}[x(u,\eta)]^2$ defined on $\tilde{\Omega}$ clearly satisfies the measurability requirements (3.5) and (3.6). From the Schwartz inequality and (4.3)

$$\int_0^T |x(u,\eta)x(u,\tilde{\eta}) - \tfrac{1}{2}[x(u,\eta)]^2|du$$

$$(4.7) \qquad \leq \int_0^T |x(u,\eta)x(u,\tilde{\eta})| + \tfrac{1}{2}\int_0^T [x(u,\eta)]^2 du$$

$$\leq \sqrt{\int_0^T [x(u,\eta)]^2 du \int_0^T [x(u,\tilde{\eta})]^2 du} + \tfrac{1}{2}\int_0^T [x(u,\eta)]^2 du < \infty \qquad \text{a.s. } \tilde{P}.$$

Thus

$$(4.8) \qquad x(u,\eta)x(u,\tilde{\eta}) - \tfrac{1}{2}[x(u,\eta)]^2 \in \tilde{\mathcal{M}}_1$$

and from (4.4) the process $\zeta(t)$ has a differential (3.106) on $\tilde{\Omega}$ where

$$(4.9) \qquad \begin{aligned} a(t,\eta,\tilde{\eta},w) &= x(t,\eta)x(t,\tilde{\eta}) - \tfrac{1}{2}[x(t,\eta)]^2 \\ b(t,\eta,\tilde{\eta},w) &= x(t,\eta) . \end{aligned}$$

We shall apply Lemma 3.7 with $n = 1$ and

$$(4.10) \qquad \Gamma(x) = e^x .$$

Thus from (3.113) $\zeta_1(t)$ has a stochastic differential

$$(4.11) \qquad d\zeta_1(t) = a_1(t)dt + b_1(t)dw(t)$$

where from (3.114), (3.115) and (4.9)

$$(4.12) \qquad a_1(t) = e^{\zeta(t)}(x(t)\tilde{x}(t) - \tfrac{1}{2}x^2(t)) + \tfrac{1}{2}e^{\zeta(t)}x^2(t) = e^{\zeta(t)}x(t)\tilde{x}(t)$$

and

$$(4.13) \qquad b_1(t) = e^{\zeta(t)}x(t) .$$

All processes are defined on $\tilde{\Omega}$ and we use the notation $\tilde{x}(t)$ to indicate $\tilde{x}(t,\eta,\tilde{\eta},w) \equiv x(t,\tilde{\eta})$.

Next we shall give a differential for the process $\zeta_2(t)$ on Ω

defined by

$$(4.14) \qquad \zeta_2(t,\tilde{\eta},w) = \int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)P_X(d\eta) .$$

First, however, we shall prove three lemmas which will be required to establish the conditions (3.18), (3.73) and (3.74) needed in the application of Lemmas 3.2 and 3.6.

Lemma 4.2 *Let* $x(t,\eta)$ *be square integrable* a.s. P_X. *Then for* $0 \le s \le u \le T$

$$(4.15) \qquad \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)} e^{4[\zeta(u)-\zeta(s)]}\right] \le E\left[e^{16\int_s^u x^2(\tau)d\tau}\right]$$

provided the right side is finite.

Proof: Theorem 2.1 applies since it is assumed that $x(t,\eta)$ is jointly measurable and square integrable a.s. P_X. From (2.14) of Theorem 2.1 and definitions (4.4), (4.5) and (4.14)

$$(4.16) \qquad 0 < \zeta_2(s) < \infty \qquad \text{a.s. } P .$$

We shall first look at the conditional expectation of the left side of (4.15) with respect to the σ-field $\tilde{\tilde{\mathcal{F}}}_s$. From the definition of $\zeta_1(s)$ and $\zeta(s)$ it is clear that $\zeta_1(s)$ is $\tilde{\tilde{\mathcal{F}}}_s$-measurable for all $0 \le s \le T$. From (4.14) and (4.2) $\zeta_2(s)$ is $\tilde{\mathcal{F}}_s$-measurable and as a process on $\tilde{\Omega}$, $\zeta_2(s)$ is $\tilde{\tilde{\mathcal{F}}}$-measurable. Thus

$$(4.17) \qquad \tilde{E}^{\tilde{\tilde{\mathcal{F}}}_s}\left[\frac{\zeta_1(s)}{\zeta_2(s)} e^{4(\zeta(u)-\zeta(s))}\right] = \frac{\zeta_1(s)}{\zeta_2(s)} \tilde{E}^{\tilde{\tilde{\mathcal{F}}}_s}\left[e^{4(\zeta(u)-\zeta(s))}\right] \qquad \text{a.s. } \tilde{P} .$$

From (4.4) and (iii) of Lemma 3.3

$$(4.18) \qquad \zeta(u)-\zeta(s) = \int_s^u x(\tau,\eta)dw(\tau) + \int_s^u x(\tau,\eta)x(\tau,\tilde{\eta})d\tau - \tfrac{1}{2}\int_s^u [x(\tau,\eta)]^2 d\tau \qquad \text{a.s. } \tilde{P}.$$

From Dynkin ([4], Theorem 7.3, p. 234)

$$(4.19) \qquad \tilde{E}^{\tilde{\mathfrak{F}}_s}\left[e^{4\int_s^u x(\tau,\eta)dw(\tau)}\right] = e^{8\int_s^u [x(\tau,\eta)]^2 du} .$$

Thus from (4.18), (4.19) and some elementary inequalities,

$$\tilde{E}^{\tilde{\mathfrak{F}}_s}\left[e^{4(\zeta(u)-\zeta(s))}\right] = e^{8\int_s^u [x(\tau,\eta)]^2 d\tau + 4\int_s^u x(\tau,\eta)x(\tau,\tilde{\eta})d\tau - 2\int_s^u [x(\tau,\eta)]^2 d\tau}$$

$$\le e^{6\int_s^u [x(\tau,\eta)]^2 d\tau + 4\left|\int_s^u x(\tau,\eta)x(\tau,\tilde{\eta})d\tau\right|}$$

$$(4.20) \qquad \le e^{6\int_s^u x^2(\tau,\eta)d\tau + 4\sqrt{(\int_s^u [x(\tau,\eta)]^2 d\tau)(\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau)}}$$

$$\le e^{6\int_s^u x^2(\tau,\eta)d\tau + 2[\int_s^u [x(\tau,\eta)]^2 d\tau + \int_s^u [x(\tau,\tilde{\eta})]^2 d\tau]}$$

$$\le e^{8\int_s^u [x(\tau,\eta)]^2 d\tau + 8\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau} .$$

Taking expectations of (4.17) and using (4.20) we have

$$(4.21) \qquad \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{4(\zeta(u)-\zeta(s))}\right] \le \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{8\int_s^u x^2(\tau,\eta)d\eta + 8\int_s^u x^2(\tau,\tilde{\eta})d\tau}\right] .$$

The Schwartz inequality applied to the right hand of the above inequality gives

$$(4.22) \qquad \begin{aligned} &\tilde{E}\left[\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} e^{8\int_s^u x^2(\tau)d\eta}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} e^{8\int_s^u \tilde{x}(\tau)d\tau}\right] \\ &\le \left\{\tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{16\int_s^u x^2(\tau,\eta)d\eta}\right] \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{16\int_s^u x^2(\tau,\tilde{\eta})d\tau}\right]\right\}^{\frac{1}{2}} . \end{aligned}$$

Examining the second factor on the right side of (4.22), from the definition of $\zeta_2(s)$ (4.14) and the Fubini Theorem for non-negative functions

$$\tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{16\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau}\right]$$

(4.23) $$= \int_W \int_{\tilde{\Omega}_X} \frac{e^{16\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau}}{\zeta_2(s,\eta,\tilde{\eta},w)} \left[\int_{\Omega_X} \zeta_1(s,\eta,\tilde{\eta},w) P_X(d\eta)\right] P_X(d\tilde{\eta}) P_W(dw)$$

$$= \int_W \int_{\tilde{\Omega}_X} e^{16\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau} P_X(d\tilde{\eta}) P_W(dw) = E\left[e^{16\int_s^u x^2(\tau) d\tau}\right].$$

From (2.15) of Theorem 2.1 for

(4.24) $$g(\eta) = e^{16\int_s^u [x(\tau,\eta)]^2 d\tau},$$

(4.25) $$\int_{\Omega_X} \frac{\zeta_1(s,\eta,\tilde{\eta},w)}{\zeta_2(s,\eta,\tilde{\eta},w)}\, e^{16\int_s^u [x(\tau,\eta)]^2 d\tau}\, P_X(d\eta)$$

$$= E^t\left[e^{16\int_s^u [x(\tau,\tilde{\eta})]^2 d\tau}\right](\tilde{\eta},w) \qquad \text{a.s. } P.$$

Taking expectations of (4.25) with respect to P,

(4.26) $$\tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{16\int_s^u [x(\tau,\eta)]^2 d\tau}\right] = E\left[e^{16\int_s^u x^2(\tau) d\tau}\right].$$

The assertion (4.15) of the lemma then follows from (4.21), (4.22), (4.23) and (4.26).

Lemma 4.3 *Let* $y(t,\eta)$, $0 \le t \le T$, *be a jointly measurable process defined on* $(\Omega_X, \mathcal{B}_X, P_X)$ *and let* $x(t,\eta)$ *be square integrable* a.s. P_X. *Then for* $0 \le s < t \le T$

$$\tilde{E}\left[\frac{\left\{\int_s^t E^{\tilde{\mathcal{F}}_s}[y^2(u)e^{2\zeta(u)}]du\right\}^{\frac{1}{2}}}{\zeta_2(s)}\right]$$

(4.27)

$$\leq \left\{\left[\int_s^t E(y^4(u))du\right]\left[\int_s^t E\left(e^{16\int_s^u x^2(\tau)d\tau}\right)du\right]\right\}^{\frac{1}{4}}$$

provided the right side is finite.

Proof: Since $\zeta(s)$ and hence $e^{2\zeta(s)}$ are $\tilde{\mathcal{F}}_s$-measurable

$$\tilde{E}\left[\frac{\left\{\int_s^t E^{\tilde{\mathcal{F}}_s}[y^2(u)e^{2\zeta(u)}]du\right\}^{\frac{1}{2}}}{\zeta_2(s)}\right]$$

(4.28)

$$= \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{-\zeta(s)}\left\{\int_s^t E^{\tilde{\mathcal{F}}_s}(y^2(u)e^{2\zeta(u)})du\right\}^{\frac{1}{2}}\right]$$

$$= \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\left\{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}(y^2(u)e^{2(\zeta(u)-\zeta(s))})du\right\}^{\frac{1}{2}}\right].$$

Recalling the definition of $\zeta_2(s)$ as used in (4.23) and the fact that $\zeta_1(s)/\zeta_2(s)$ is $\tilde{\mathcal{F}}_s$-measurable we obtain from the Schwartz inequality and the Fubini Theorem

$$\tilde{E}\left[\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}}\left\{\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}(y^2(u)e^{2(\zeta(u)-\zeta(s))})du\right\}^{\frac{1}{2}}\right]$$

$$\leq \left\{\tilde{E}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)\tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\int_s^t \tilde{E}^{\tilde{\mathcal{F}}_s}(y^2(u)e^{2(\zeta(u)-\zeta(s))})du\right]\right\}^{\frac{1}{2}}$$

(4.29)

$$= \left\{\int_s^t \tilde{E}\,\tilde{E}^{\tilde{\mathcal{F}}_s}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\, y^2(u)e^{2(\zeta(u)-\zeta(s))}\right)du\right\}^{\frac{1}{2}}$$

$$= \left\{\int_s^t \tilde{E}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\, y^2(u)e^{2(\zeta(u)-\zeta(s))}\right)du\right\}^{\frac{1}{2}}.$$

By the Schwartz inequality again,

$$
\left\{\int_s^t \tilde{E}\left[\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} y^2(u)\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} e^{2(\zeta(u)-\zeta(s))}\right]du\right\}^{\frac{1}{2}}
$$

(4.30)

$$
\leq \left\{\int_s^t \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, y^4(u)\right] du\right\}^{\frac{1}{4}} \left\{\int_s^t \tilde{E}\left[\frac{\zeta_1(s)}{\zeta_2(s)}\, e^{4(\zeta(u)-\zeta(s))}\right] du\right\}^{\frac{1}{4}} .
$$

From (2.15) of Theorem 2.1 as used in (4.25) and (4.26)

$$
(4.31) \qquad \int_s^t \tilde{E}\left(\frac{\zeta_1(s)}{\zeta_2(s)}\, y^4(u)\right) dy = \int_s^t E(y^4(u))du .
$$

The function $g(\eta) = [y(u,\eta)]^4$ here has finite expectation since it is assumed that the right side of (4.27) is finite. The result (4.27) then follows from (4.28), (4.29), (4.30), (4.31) and Lemma 4.2.

Lemma 4.4 Let $x(\tau,\eta)$ and $y(\tau,\eta)$ be jointly measurable and let $x(t,\eta)$ be square integrable a.s. P_X. Then for $0 \leq s < t \leq T$

$$
E\left\{\frac{1}{[\zeta_2(s,\tilde{\eta},w)]^2} \int_s^t [\int_{\Omega_X} |y(u,\eta)| e^{\zeta(u,\eta,\tilde{\eta},w)} P_X(d\eta)]^2 du\right\}
$$

(4.32)

$$
\leq [\int_s^t E[y^4(u)]du]^{\frac{1}{2}} \left[\int_s^t E\left(e^{16\int_s^u x^2(\tau)d\tau}\right) du\right]^{\frac{1}{2}}
$$

provided the right side is finite.

Proof: By the Schwartz inequality,

$$
[\int_{\Omega_X} |y(u,\eta)| e^{\zeta(u)} P_X(d\eta)]^2
$$

$$
(4.33) \qquad = [\int_{\Omega_X} |y(u,\eta)| e^{\frac{1}{2}\zeta(s)}\, e^{\frac{1}{2}\zeta(s)+(\zeta(u)-\zeta(s))} P_X(d\eta)]^2
$$

$$
\leq [\int_{\Omega_X} y^2(u) e^{\zeta(s)} P_X(d\eta)][\int_{\Omega_X} e^{\zeta(s)+2(\zeta(u)-\zeta(s))} P_X(d\eta)] .
$$

Hence,

$$E\left[\int_s^t \frac{\left[\int_{\Omega_X} |y(u)| e^{\zeta(u)} P_X(d\eta)\right]^2}{(\zeta_2(s))^2} du\right]$$

$$(4.34) \quad \leq E \int_s^t \left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} y^2(u) P_X(d\eta)\right]\left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} e^{2(\zeta(u)-\zeta(s))} P_X(d\eta)\right] du$$

$$\leq \left\{E \int_s^t \left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} y^2(u) P_X(d\eta)\right]^2 du\right\}^{\frac{1}{2}} \left\{E \int_s^t \left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} e^{2(\zeta(u)-\zeta(s))} P_X(d\eta)\right]^2 du\right\}^{\frac{1}{2}}.$$

Each factor will be examined separately. By the Schwartz inequality and (4.14)

$$\left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} y^2(u) P_X(d\eta)\right]^2 = \left[\int_{\Omega_X} \left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} \left(\frac{\zeta_1(s)}{\zeta_2(s)}\right)^{\frac{1}{2}} y^2(u) P_X(d\eta)\right]^2$$

$$(4.35) \quad \leq \int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} y^4(u) P_X(d\eta) \quad .$$

Integrating and taking expectations of (4.35), then from (4.31)

$$E \int_s^t \left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} y^2(u) P_X(d\eta)\right]^2 du$$

$$(4.36) \quad \leq \int_s^t \tilde{E}\left[\left(\frac{\zeta_1(s)}{\zeta_2(s)}\right) y^4(u)\right] du = \int_s^t E[y^4(u)] du \ .$$

Using the same technique on the second factor on the right side of (4.34) and then Lemma 4.2, we find

$$E \int_s^t \left[\int_{\Omega_X} \frac{\zeta_1(s)}{\zeta_2(s)} e^{2(\zeta(u)-\zeta(s))} P_X(d\eta)\right]^2 du$$

$$(4.37) \quad \leq E \int_s^t \int_{\Omega_X} \left(\frac{\zeta_1(s)}{\zeta_2(s)} e^{4(\zeta(u)-\zeta(s))}\right) P_X(d\eta) du$$

$$\leq \int_s^t E\left[e^{16\int_s^u x^2(\tau) d\tau}\right] du \ .$$

The result (4.23) then follows from (4.34), (4.36) and (4.37).

Lemma 4.5 Let $x(t,\eta)$ and $y(t,\eta)$ be jointly measurable processes which satisfy

$$(4.38)\qquad \int_0^T x^2(t)dt < \infty \qquad \text{a.s. } P_X ,$$

$$(4.39)\qquad \int_0^T E[y^4(t)]dt < \infty ,$$

and let there exist $\Delta > 0$ such that

$$(4.40)\qquad E\left[e^{16\int_t^{t+\Delta} x^2(u)du}\right] < \infty ,$$

for all $0 \le t \le T-\Delta$. Assume that a process ζ_1^y on $(\tilde{\Omega},\tilde{\mathcal{B}},\tilde{P})$ satisfies

$$(4.41)\qquad \int_{\Omega_X} |\zeta_1^y(t,\eta,\tilde{\eta},w)|P_X(d\eta) < \infty \qquad \text{a.s. } P$$

and has a stochastic differential

$$(4.42)\qquad d\zeta_1^y(t) = [a_0(t)+\zeta_1(t)y(t)\tilde{x}(t)]dt+\zeta_1(t)y(t)dw(t) \qquad 0 \le t \le T$$

where the process $a_0(t)$ satisfies

$$(4.43)\qquad a_0 \in \tilde{\mathcal{M}}_1$$

and

$$(4.44)\qquad \int_0^T [\int_{\Omega_X} |a_0(t,\eta,\tilde{\eta},w)|P_X(d\eta)]dt < \infty \qquad \text{a.s. } P .$$

Then the process ζ_2^y defined on $(\Omega,\mathcal{B},P)$ by

$$(4.45)\qquad \zeta_2^y(t,\tilde{\eta},w) = \int_{\Omega_X} \zeta_1^y(t,\eta,\tilde{\eta},w)P_X(d\eta)$$

has a stochastic differential

$$d\zeta_2^y(t,\tilde{\eta},\omega)$$

$$(4.46)\qquad = [\int_{\Omega_X} a_0(t,\eta,\tilde{\eta},w)P_X(d\eta) + \int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)y(t,\eta)x(t,\tilde{\eta})P_X(d\eta)]dt$$

$$+ [\int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)y(t,\eta)P_X(d\eta)]dw(t) \qquad 0 \le t \le T .$$

Proof: Since by assumption $a_0(t)$ and $a_0(t)+\zeta_1(t)y(t)\tilde{x}(t) \in \tilde{\mathcal{M}}_1$ it follows that

$$(4.47) \qquad a(t) = \zeta_1(t)y(t)\tilde{x}(t) \in \tilde{\mathcal{M}}_1 .$$

It is also assumed that $\zeta_1(t)y(t) \in \tilde{\mathcal{M}}_2$. We shall show that Lemmas 3.2 and 3.6 can be applied to a_0, a and $b(u) = \zeta_1(u)y(u)$, all restricted to the interval $s \le u \le t$, where $0 \le s < t \le s+\Delta \le T$. From (4.43) and (4.44) a_0 satisfies (3.18) of Lemma 3.2. By the Schwartz inequality

$$\int_s^t [\int_{\Omega_X} |\zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)x(u,\tilde{\eta})|P_X(d\eta)]du$$

$$(4.48) \qquad \le \int_s^t [\int_{\Omega_X} |\zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)|P_X(d\eta)]|x(u,\tilde{\eta})|du$$

$$\le \left\{\int_s^t [\int_{\Omega_X} |\zeta_1(x,\eta,\tilde{\eta},w)y(u,\eta))P_X(d\eta)]^2du \cdot \int_s^t x^2(u,\eta)du\right\}^{\frac{1}{2}}. \qquad \text{a.s. } P.$$

Since

$$(4.49) \qquad E\left[e^{16\int_s^u x^2(\tau)d\tau}\right]$$

is clearly an increasing function of u, for $s \le t \le s+\Delta$

$$(4.50) \qquad \int_s^t E\left[e^{16\int_s^u x^2(\tau)d\tau}\right]du \le \Delta\, E\left[e^{16\int_s^{s+\Delta} x^2(\tau)d\tau}\right] < \infty$$

by assumption (4.40). From (4.50), (4.39) and Lemma 4.4 we have

$$(4.51) \qquad E\left\{\frac{\int_s^t [\int_{\Omega_X} |y(u,\eta)|\zeta_1(u,\eta,\tilde{\eta},w)P_X(d\eta)]^2du}{\zeta_2^2(s)}\right\} < \infty .$$

From (4.16) it then follows that $\int_s^t [\int_{\Omega_X} |y(u,\eta)|\zeta_1(u,\eta,\tilde{\eta},w)P_X(d\eta)]^2du$ is finite a.s. P. This, together with assumption (4.38) shows the finiteness a.s. P of the right hand side of (4.48) and we conclude that the process $a(t)$ given by (4.47) satisfies condition (3.18) of Lemma 3.2. From (4.39), (4.40) and (4.50) the right side of the imequality (4.27) of Lemma 4.3 is seen to be finite. Hence

$$\tilde{E}\left[\frac{\{\int_s^t E^{\tilde{\mathcal{F}}_s}(y^2(u)\zeta_1^2(u)du\}^{\frac{1}{2}}}{\zeta_2(s)}\right] < \infty . \tag{4.52}$$

Finally from (4.16), (4.51) and (4.52) it follows that conditions (3.73) and (3.74) of Lemma 3.6 are satisfied for the process $b(u) = \zeta_1(u)y(u)$. We are now in a position to apply Lemmas 3.2 and 3.6 and to conclude that

$$\int_{\Omega_X} a_0(t,\eta,\tilde{\eta},w)P_X(d\eta) \quad \epsilon \quad \mathcal{M}_1 \tag{4.53}$$

$$\int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)y(t,\eta)x(t,\tilde{\eta})P_X(d\eta) \quad \epsilon \quad \mathcal{M}_1 \tag{4.54}$$

and

$$\int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)y(t,\eta)P_X(d\eta) \quad \epsilon \quad \mathcal{M}_2 . \tag{4.55}$$

From (4.53) and (4.54) it follows that the sum of these two terms also belongs to $\mathcal{M}_1$.

By assumption (4.42) for $0 \le s < t \le T$

$$\zeta_1^y(t)-\zeta_1^y(s) = \int_s^t [a_0(u)+\zeta_1(u)y(u)\tilde{x}(u)]du+\int_s^t \zeta_1(u)y(u)dw(u) \qquad \text{a.s. } \tilde{P} . \tag{4.56}$$

From the finiteness a.s. P of integrals assured by assumption (4.41)

$$\begin{aligned} &\int_{\Omega_X}[\zeta_1^y(t,\eta,\tilde{\eta},w)-\zeta_1^y(s,\eta,\tilde{\eta},w)]P_X(d\eta) \\ &= \int_{\Omega_X}\zeta_1^y(t,\eta,\tilde{\eta},w)P_X(d\eta)-\int_{\Omega_X}\zeta_1^y(u,\eta,\tilde{\eta},w)P_X(d\eta) = \zeta_2^y(t)-\zeta_2^y(s) \\ &\qquad\qquad \text{a.s. } P . \end{aligned} \tag{4.57}$$

From Lemma 3.2

$$\int_{\Omega_X} [\int_s^t a_0(u,\eta,\tilde{\eta},w)du]P_X(d\eta) = \int_s^t [\int_{\Omega_X} a_0(u,\eta,\tilde{\eta},w)P_X(d\eta)]du \qquad \text{a.s. } P , \tag{4.58}$$

$$\int_{\Omega_X} [\int_s^t \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)x(u,\tilde{\eta})du]P_X(d\eta)$$

(4.59)
$$= \int_s^t [\int_{\Omega_X} \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)x(u,\tilde{\eta})P_X(d\eta)]du \qquad \text{a.s. } P,$$

and these integrals are finite a.s. P. From (3.76) of Lemma 3.6

$$\int_{\Omega_X} [\int_s^t \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)dw(u)]P_X(d\eta)$$

(4.60)
$$= \int_s^t [\int_{\Omega_X} \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)P_X(d\eta)]dw(u) \qquad \text{a.s. } P$$

and the integrals (4.60) are finite a.s. P. Thus from (4.56) - (4.60), for $0 \le s < t \le s+\Delta \le T$

$$\zeta_2^y(t) - \zeta_2^y(s) = \int_s^t [\int_{\Omega_X} a_0(u,\eta,\tilde{\eta},w)P_X(d\eta)$$

(4.61)
$$+ \int_{\Omega_X} \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)x(u,\tilde{\eta})P_X(d\eta)]du$$

$$+ \int_s^t [\int_{\Omega_X} \zeta_1(u,\eta,\tilde{\eta},w)y(u,\eta)P_X(d\eta)]dw(u) \qquad \text{a.s. } P\,.$$

Since (4.61) holds for all $s \le t \le s+\Delta$, where $0 < \Delta$, from the linearity of the integrals it is easily seen that (4.61) holds for all $0 \le s < t \le T$ and hence that (4.46) is satisfied.

<u>Lemma</u> 4.6 <u>If the process</u> $x(t,\eta)$, $0 \le t \le T$ <u>satisfies</u>

(4.62)
$$\int_0^T E[x^4(t)]dt < \infty$$

<u>and there exists</u> $\Delta > 0$ <u>such that</u>

(4.63)
$$E\left[e^{16\int_s^{s+\Delta} x^2(u)du}\right] < \infty ,$$

<u>for all</u> $0 \le s \le T-\Delta$. <u>Then the process</u> $\zeta_2(t)$ <u>defined on</u> $(\Omega,\mathcal{B},P)$ <u>by</u> (4.14) <u>has a differential</u>

$$d\zeta_2(t,\tilde{\eta},w) = [\int_{\Omega_X}\zeta_1(t,\eta,\tilde{\eta},w)x(t,\eta)P_X(d\eta)]x(t,\tilde{\eta})dt$$

(4.64)

$$+ [\int_{\Omega_X}\zeta_1(t,\eta,\tilde{\eta},w)x(t,\eta)P_X(d\eta)]dw(t) \qquad 0 \leq t \leq T.$$

Proof: In Lemma 4.5 let

(4.65) $$a_0(t) = 0 ,$$

(4.66) $$y(t,\eta) = x(t,\eta)$$

and

(4.67) $$\zeta_1^y(t) = \zeta_1(t) .$$

Then from Lemma 4.1, $\zeta_1(t)$ has a differential (4.42). From (4.14) and (4.16), condition (4.41) is also satisfied. The result (4.64) then follows from (4.46) of Lemma 4.5.

5. <u>Markov-Processes with Generalized Infinitesimal Generator</u>. In this section additional restrictions will be imposed on the system process $x(t,\eta)$ $0 \le t \le T$, $\eta \in \Omega$. We will say that a Markov process $x(t,\eta)$ has jointly measurable transition probabilities provided it has regular transition probabilities

$$(5.1) \qquad P[x(t) \in B | x(s) = x] = P(s,x;t,B)$$

(x real and $0 \le s \le t \le T$) which are jointly measurable in $(s,x;t)$ for all Borel sets of the real line $B \in \mathcal{B}_R$. We will be concerned with the generalized semi-group (in the sense of Loeve [14], p. 568) of operators P_s^t, $0 \le s \le t \le T$, defined by

$$(5.2) \qquad (P_s^t f)(x) = \int_{-\infty}^{\infty} f(y)P(s,x;tdy) \qquad -\infty < x < \infty ,$$

on the Banach space with sub norm of bounded measurable functions of a real variable; i.e., for

$$(5.3) \qquad f \in B(R, \mathcal{B}_R) .$$

The generalized semi-group property, which corresponds to the Chapman-Kolmogorov equation for Markov processes, is given by

$$(5.4) \qquad P_s^t = P_s^u P_u^s \qquad \text{for } 0 \le s \le u \le t \le T.$$

It will be assumed that the semi-group P_s^t has a generalized infinitesimal generator $\mathcal{G}_t$, $0 \le t \le T$, defined on a domain $\mathcal{D} \subset B(R, \mathcal{B}_R)$. More precisely, it is assumed that for $0 \le t \le T$, $\mathcal{G}_t$ is a linear operator with domain $\mathcal{D}$ and range $B(R, \mathcal{B}_R)$ which satisfies

$$(5.5) \qquad \sup_{\substack{-\infty<x<\infty \\ 0\le t\le T}} |\mathcal{G}_t f)(x)| < \infty$$

and

$$\text{(5.6)} \qquad \sup_{\substack{-\infty<x<\infty \\ 0\le t\le T}} \left| (G_t f)(x) - \left[\frac{(P_t^{[t+h]}f-f)}{h}\right](x)\right| \to 0 \qquad \text{as } h\downarrow 0$$

for all $f \in \mathcal{D}$, where

$$\text{(5.7)} \qquad [t+h] = \begin{cases} t+h & \text{if } t+h \le T \\ T & \text{if } t+h > T \end{cases} .$$

When (5.5) and (5.6) are satisfied for $f \in \mathcal{D}$, G_t will be said to be the strong generalized infinitesimal generator of the process x. The domain $\mathcal{D}$ of G_t will be extended later.

The following lemma is suggested and a sketch of its proof is given in Dynkin ([3], 4.6, p. 102).

<u>Lemma</u> 5.1 <u>Let</u> x <u>be a measurable Markov process with jointly measurable transition probabilities</u> (5.1) <u>and strong generalized infinitesimal generator</u> G_t <u>on</u> $\mathcal{D}$. <u>Then for</u> $f \in \mathcal{D}$

$$\text{(5.8)} \qquad (G_t f)(x)$$

<u>is jointly measurable in</u> (t,x) <u>and</u>

$$\text{(5.9)} \qquad (P_s^t f-f)(x) = \int_s^t (P_s^u G_u f)(x)du \qquad (0 \le s \le t \le T,\ -\infty < x < \infty)$$

<u>where the operators</u> P_s^t <u>are defined by</u> (5.2).

Proof: First extend the process $x(\tau)$, $0 \le \tau \le T$ to the range $0 \le \tau < \infty$ by defining

$$\text{(5.10)} \qquad x(\tau,\eta) = x(T,\eta) \qquad \text{for } \tau > T.$$

The generalized semi-group P_s^t is thus extended to $0 \le s \le t < \infty$ by letting

$$\text{(5.11)} \qquad P_s^t = P_s^T \qquad \text{for } s \le T < t$$

and

(5.12) $$P_s^t = I \qquad \text{for } T \le s \le t .$$

Then following Dynkin [3] define operators S_τ, $0 \le \tau < \infty$ on

(5.13) $$\tilde{B} = B[[0,\infty) \times R, \mathcal{B}_{[0,\infty)} \times \mathcal{B}_R] ,$$

the Banach space of bounded measurable functions $\tilde{f}(t,\chi)$ of two variables $0 \le t < \infty$, $-\infty < x < \infty$, by

(5.14) $$(S_\tau \tilde{f})(t,x) = \int_{-\infty}^{\infty} \tilde{f}(t+\tau,y)P(t,x;t+\tau,dy) .$$

Thus, from the assumption that $P(s,x;\tau,B)$ is jointly measurable in (s,x,t), it follows from (5.14) that $(S_\tau \tilde{f})(t,x)$ is measurable in (t,x) and hence that

(5.15) $$S_\tau \tilde{f} \in \tilde{B} .$$

It is shown in [3] that S_τ defined by (5.14) is a semi-group. In [4] (Theorem 1.3C, p. 23) it is also shown that

(5.16) $$(S_h\tilde{f}-\tilde{f})(t,x) = \int_0^h (S_\tau \tilde{G}\tilde{f})(t,x)d\tau$$

for all $\tilde{f} \in \tilde{\mathcal{D}}$ where $\tilde{G}$ is the infinitesimal generator of the semi-group S_τ with domain $\tilde{\mathcal{D}}$. That is, $\tilde{\mathcal{D}}$ is the set of all functions $\tilde{f} \in \tilde{B}$ on which $\tilde{G}\tilde{f} \in \tilde{B}$ can be found which satisfies

(5.17) $$\sup_{\substack{-\infty<x<\infty \\ 0\le t\le\infty}} \left|\left(\tilde{G}\tilde{f} - \frac{(S_h\tilde{f}-\tilde{f})}{h}\right)(t,x)\right| \to 0 \qquad \text{as } h \to 0 .$$

Consider the family of functions $\tilde{\mathcal{D}}_0$ defined by

(5.18) $$\tilde{\mathcal{D}}_0 = \{\tilde{f} \mid \tilde{f}(t,x) = f(x) \ \begin{matrix} 0 \le t \le \infty \\ -\infty < x < \infty \end{matrix} \ \text{where } f \in \mathcal{D}\}$$

and the operator $\tilde{G}_0$ defined on $\tilde{\mathcal{D}}_0$ by

(5.19) $$(\tilde{G}_0\tilde{f})(t,x) = \begin{cases} (G_t f)(x) & \text{for } 0 \le t < T \\ 0 & \text{for } T < t \end{cases} ,$$

where $\tilde{f}(t,x) = f(x)$ as in (5.18). It will be shown that $\tilde{\mathcal{D}}_0 \subset \tilde{\mathcal{D}}$ and that on $\tilde{\mathcal{D}}_0$

$$\tilde{G}\tilde{f} = \tilde{G}_0\tilde{f} \, . \tag{5.20}$$

First, it will be shown that (5.8) of the lemma is measurable in (t,x) and hence that (5.19) is measurable with respect to $\mathcal{B}_{[0,\infty)} \times \mathcal{B}_R$. From the measurability assumptions on (5.1) and the definition (5.2) it is clear that $(P_t^{[t+h]}f)(x)$ is jointly measurable in (t,x) for h fixed. Thus from (5.6) $((G_t f)(x)$ is a point-wise limit of functions measurable in (t,x) and hence is itself measurable. From the assumption (5.5) for $f \in \mathcal{D}$, $\tilde{G}_0\tilde{f}$ is also bounded for $\tilde{f} \in \tilde{\mathcal{D}}_0$ and hence

$$\tilde{G}_0 \tilde{f} \in \tilde{B} \, . \tag{5.21}$$

It is sufficient then to show that (5.17) holds for $\tilde{f} \in \tilde{\mathcal{D}}_0$ and $\tilde{G} = \tilde{G}_0$ given by (5.19). From (5.2), (5.7), (5.11), (5.12), (5.14) and (5.18)

$$(S_h\tilde{f})(t,x) = \begin{cases} (P_t^{[t+h]}f)(x) & \text{for } 0 \le t \le T \\ f(x) & \text{for } T < t. \end{cases} \tag{5.22}$$

It is easily seen then that (5.17) follows from the assumption that (5.6) holds for $f \in \mathcal{D}$. Thus the assertion (5.20) has been established and hence (5.16) holds for $\tilde{G}_0$ and $\tilde{f}$ given by (5.19) and (5.18). For $f \in \mathcal{D}$, from (5.22) and (5.18)

$$(S_h\tilde{f}-\tilde{f})(s,x) = (P_s^{s+h}f-f)(x), \qquad s+h \le T \tag{5.23}$$

and from (5.19), (5.14) and (5.2)

$$(S_\tau(\tilde{G}_0\tilde{f})(s,x) = (P_s^{s+\tau}G_{s+\tau}f)(x), \qquad s+\tau \le T \, . \tag{5.24}$$

Thus the result (5.9) of the lemma then follows from (5.16), (5.23) and (5.24) by letting $t = s+h$ and $u = s+\tau$.

The domain $\mathcal{D}$ of the generalized infinitesimal generator $\{G_t\}$ often turns out to be not quite large enough for our purposes, especially in applications where we need to obtain a stochastic differential equation for $E^t[f(x_t)]$ for certain unbounded functions f. The widest and most natural class say $\mathcal{D}^*$ of functions for which we can derive the basic stochastic differential equation is defined as follows. Let $\mathcal{D}^*$ be the class of Borel measurable functions f which satisfy

$$(5.25) \qquad E|f(x_t)| < \infty \qquad \text{for each } t;$$

there exists a (t,x) Borel measurable function $(G_t^* f)(x)$ (G_t^* may be regarded as an operator acting on f) such that

$$(5.26) \qquad \int_0^T E|(G_t^* f)(x_t)|dt < \infty ,$$

and for $0 \le s < t \le T$

$$(5.27) \qquad (P_s^t f)(x(s))-f(x(s)) = \int_s^t (P_s^u G_u^* f)(x(s))du \qquad \text{a.s. } P_X.$$

When (5.25) - (5.27) are satisfied for $f \in \mathcal{D}^*$, we shall say that G_t^* is the extended generator of the process $x(t)$.

Clearly $\mathcal{D}^*$ contains $\mathcal{D}$ since for any $f \in \mathcal{D}$ from Lemma 5.1, (5.27) holds with $G_t^* = G_t$ and $(G_t f)(x)$ is measurable. Properties (5.25) and (5.26) follow from the boundedness of f and (5.5).

Sometimes, as in the examples in Section 8, it is easy to verify derectly that a function f lies in $\mathcal{D}^*$. However, since it is the strong infinitesimal generator of a Markov process which has received the most attention in the literature, we will give a set of simple sufficient conditions that will tell us when a function, which is a point-wise limit of functions in $\mathcal{D}$, belongs to $\mathcal{D}^*$.

<u>Lemma</u> 5.2 <u>If</u> $f_n \in \mathcal{D}$ <u>satisfy the following conditions</u>: <u>for each</u> x

$$(5.28) \qquad f_n(x) \to f(x),$$

$$(5.29) \qquad |f_n(x)| \le H(x),$$

$$(5.30) \qquad E[H(x(t))] < \infty \qquad 0 \le t \le T,$$

$$(5.31) \qquad (G_t f_n)(x) \to (G_t^* f)(x) \qquad 0 \le t \le T,$$

$$(5.32) \qquad (G_t f_n)(x) \le M(t,f)(x) \qquad 0 \le t \le T,$$

<u>and</u>

$$(5.33) \qquad \int_0^T E[M(t,f)(x(t))]dt < \infty,$$

<u>then</u> $f \in \mathcal{D}^*$.

Proof: Property (5.25) follows from (5.28) - (5.30), and (5.26) follows from (5.31) - (5.33). From (5.33)

$$(5.34) \qquad E[M(u,f)(x(u))] < \infty \qquad \text{a.e. } \mu_{[0,T]}.$$

For u such that (5.35) holds, from (5.31) and (5.32), all the $(G_u^* f_n)(x(u))$ and $(G_u^* f)(x(u))$ are absolutely integrable. Thus from (5.31) and properties of conditional expectations

$$(5.35) \qquad E[G_u f_n(x(u))|x(s)] \to E[G_u^* f(x(u))|x(s)] \qquad \text{a.s. } P_X$$

and

$$(5.36) \qquad E[G_u f_n(x(u))|x(s)] \le E[M(u,f)(x(u))|x(s)] \qquad \text{a.s. } P_X.$$

Thus from the dominated convergence theorem, (5.1) and (5.2)

$$(5.37) \qquad E\left|\int_s^t (P_s^u G_u f_n)(x(s))du - \int_s^t (P_s^u G_u^* f)(x(s))du\right| \to 0 \qquad \text{as } n \to \infty.$$

Similarly, from (5.28) - (5.30), it can be shown that

$$(5.38) \qquad E|(P_s^t f_n)(x(s)) - (P_s^t f)(x(s))| \to 0 \qquad \text{as } n \to \infty$$

and

$$E[|f_n(x(s)) - f(x(s))|] \to 0 \qquad \text{as} \quad n \to \infty . \tag{5.39}$$

From Lemma 5.1

$$(P_s^t f_n)(x) - f_n(x) = \int_s^t (P_s^u G_u f)(x)\,du . \tag{5.40}$$

The result (5.27) for f then follows from (5.37) - (5.40) and the uniqueness a.s. P_X of limits in L_1 mean.

6. Ito Equation for $E^t[f(x(t))]$. In this section we will make use of the following assumptions concerning the process $x(t,\eta)$, $0 \le t \le T$, $\eta \in \Omega_X$;

(6.1) $x(t,\eta)$ is a jointly measurable Markov process,

(6.2) $x(t,\eta)$ has regular transition probabilities given by (5.1) which are jointly measurable,

(6.3) $x(t,\eta)$ has an extended generator G_t^* defined on $\mathcal{D}^*$ (i.e., (5.25), (5.26) and (5.27) are satisfied for $f \in \mathcal{D}^*$),

(6.4) $$\int_0^T E[x(t)]^4 dt < \infty$$

and

there exists $\Delta > 0$ such that

(6.5) $$E\left[e^{16\int_t^{t+\Delta} x^2(u)du}\right] < \infty$$

for all $0 \le t \le T-\Delta$.

First, we shall prove a lemma which follows easily from the definition of $\mathcal{D}^*$ given in the previous section.

Lemma 6.1 If $x(t,\eta)$ satisfies (6.1) - (6.3), then for $f \in \mathcal{D}^*$, the process

(6.6) $$(G_t^* f)(x(t,\eta)) \in \tilde{\mathcal{M}}_1$$

and the process ζ_3 defined by

(6.7) $$\zeta_3(t,\eta) = f(x(T,\eta)) - \int_t^T ((G_u f)(x(u,\eta))du$$

is jointly measurable and has a stochastic differential

(6.8) $$d\zeta_3(t) = (G_t^* f)(x(t))dt \qquad 0 \le t \le T .$$

Proof: Since $f \in \mathcal{D}^*$, $(\mathcal{G}_t^* f)(x)$ is jointly measurable in (t,x), it follows then that (6.6) is jointly measurable in (t,η). For t fixed (6.6) is measurable in η with respect to $\mathcal{B}_X$ and hence from (4.2) with respect to $\tilde{\mathcal{F}}_s$. Then (6.6) follows from (5.26) for $f \in \mathcal{D}^*$. The condition (3.109) for the existence of the stochastic differential follows from the definition (6.7) of ζ_3. From (5.26) it follows that the second term on the right of (6.7) is continuous in t a.s. P_X and hence that (6.7) is jointly measurable.

<u>Lemma</u> 6.2 <u>If</u> $x(t,\eta)$ <u>satisfies</u> (6.1) - (6.4), <u>then for</u> $f \in \mathcal{D}^*$, $\zeta_1(t)$ <u>given by</u> (4.4), (4.5) <u>and</u> ζ_3 <u>given by</u> (6.7), <u>the process</u>

$$\zeta_4(t) = \zeta_1(t)\zeta_3(t) \tag{6.9}$$

<u>has a differential</u>

$$d\zeta_4(t) = a_4(t)dt + b_4(t)dw(t) \qquad 0 \le t \le T \tag{6.10}$$

<u>where</u>

$$a_4(t) = \zeta_1(t)(\mathcal{G}_t^* f)(x(t)) + \zeta_1(t)\zeta_3(t)x(t)\tilde{x}(t) \tag{6.11}$$

and

$$b_4(t) = \zeta_1(t)\zeta_3(t)x(t). \tag{6.12}$$

Proof: We will apply Lemma 3.7 where

$$\Gamma(x) = x_1 x_3 \qquad (n = 2 \text{ and } x = (x_1,x_3)). \tag{6.13}$$

The function Γ clearly has the required continuous derivatives. From (6.1) and since (6.4) implies (4.3) Lemma 4.1 applies and ζ_1 has the differential

$$d\zeta_1(t) = a_1(t)dt + b_1(t)dw(t) \qquad 0 \le t \le T \tag{6.14}$$

where

$$a_1(t) = \zeta_1(t)x(t)\tilde{x}(t) \tag{6.15}$$

and

$$b_1(t) = \zeta_1(t)x(t) . \tag{6.16}$$

From (6.1) - (6.3), Lemma 6.1 applies so that

$$d\zeta_3(t) = a_3(t)dt + b_3(t)dw(t) \qquad 0 \le t \le T \tag{6.17}$$

where

$$a_3(t) = (G_t^* f)(x(t)) \tag{6.18}$$

and

$$b_3(t) = 0 . \tag{6.19}$$

The result (6.7) - (6.9) then follows from Lemma 3.7.

Lemma 6.3 *Let* $x(t,\eta)$ *satisfy* (6.1) - (6.5) *and let* $f \in \mathcal{D}^*$ *be such that* ζ_3 *given by* (6.7) *satisfies*

$$\int_0^T E[\zeta_3^4(t)x^4(t)]dt < \infty . \tag{6.20}$$

Then the process $\zeta_5(t)$ *defined by*

$$\zeta_5(t,\tilde{\eta},w) = \int_{\Omega_X} \zeta_4(t,\eta,\tilde{\eta},w)P_X(d\eta) \tag{6.21}$$

has a differential

$$d\zeta_5(t,\tilde{\eta},w) = [\int_{\Omega_X} a_4(t,\eta,\tilde{\eta},w)P_X(d\eta)]dt+[\int_{\Omega_X} b_4(t,\eta,\tilde{\eta},w)P_X(d\eta)]dw(t) \qquad 0 \le t \le T \tag{6.22}$$

where a_4 *and* b_4 *are given by* (6.11) and (6.12).

Proof: We will apply Lemma 4.5 to

$$y(t,\eta) = \zeta_3(t,\eta)x(t,\eta) \tag{6.23}$$

and

$$a_0(t) = \zeta_1(t)((G_u^* f)(x(t)) . \tag{6.24}$$

From Lemma 6.2 it is seen that

(6.25) $$\zeta_1^y(t) = \zeta_4(t)$$

has a differential of the form (4.42). Since $f \in \mathcal{D}^*$ from assumptions (5.25) and (5.26)

(6.26) $$E[|\zeta_3(t)|] \le E[|f(X(T))|] + \int_0^T E|((G_u^* f)(x(u))|du < \infty$$

for all $0 \le t \le T$. Thus Theorem 2.1 applies for

(6.27) $$g(\eta) = |\zeta_3(t,\eta)|$$

and

(6.28) $$E\left[\frac{1}{\zeta_2(t)} \int_{\Omega_X} \zeta_1(t)|\zeta_3(t)|P_X(d\eta)\right] = E\left[E^t|\tilde{\zeta}_3(t)|\right] = E|\zeta_3(t)|$$

which is finite from (6.26). Since $0 < \zeta_2(t) < \infty$ a.s. P it follows that

(6.29) $$\int_{\Omega_X} \zeta_1(t)|\zeta_3(t)|P_X(d\eta) < \infty \qquad \text{a.s. } P.$$

Thus (4.41) is satisfied since $\zeta_1(t) \ge 0$. The measurability properties (3.5) and (3.6) of a_0 given by (6.24) follow from those of $\zeta_1(t)$ since $((G_t^* f)(x(t))$ is $\mathcal{B}_X \times \tilde{\Omega}_X \times W \subset \tilde{\mathcal{F}}_t$ measurable for all t.

Again from assumption (5.26) for $f \in \mathcal{D}^*$

(6.30) $$E \int_0^T |(G_t^* f)(x(t))|dt < \infty .$$

Thus

(6.31) $$E[|(G_t^* f)(x(t))|] < \infty \qquad \text{a.e. } \mu_{[0,T]}$$

and for t such that (6.31) holds, Theorem 2.1 applies and hence

(6.32) $$\int_0^T E\left[\frac{1}{\zeta_2(t)} \int_{\Omega_X} \zeta_1(t)|(G_t^* f)(x(t))|P_X(d\eta)\right]dt = \int_0^T E[E^t[|((G_t^* f)(\tilde{x}(t))|]dt = \int_0^T E[|G_t^* f(x(t))|]dt .$$

Since this is finite, it follows as before that

$$(6.33) \qquad \int_{\Omega_X} \zeta_1(t)\,|(G_t^* f)(x(t))|\,P_X(d\eta) < \infty \qquad \text{a.s. } P$$

and hence that (4.44) holds.

Since from Lemma 4.1 $\zeta_1(t)$ has a stochastic differential, there is a version of $\zeta_1(t)$ which has continuous sample functions a.s. $\tilde{P}$ and hence they are bounded a.s. on the interval $[0,T]$ (see, for example, Skorokhod [17], Theorem 3, p. 21). Indicating this bound by $M(\eta,\tilde{\eta},w)$

$$(6.34) \qquad \int_0^T \zeta_1(t,\eta,\tilde{\eta},w)\,|(G_t^* f)(x(t,\eta))|\,dt \le M(\eta,\tilde{\eta},w) \int_0^T |(G_t^* f)(x(t,\eta))|\,dt \qquad \text{a.s. } \tilde{P}.$$

The second factor on the right of (6.34) is finite a.s. P_X since its expectation is finite. It follows then that (6.34) is finite a.s. $\tilde{P}$ and hence that $a_0 \in \tilde{\mathcal{M}}$. Thus the conditions (4.38) - (4.44) of Lemma 4.5 all hold, the lemma applies and the result (6.22) follows from (4.46) of Lemma 4.5.

The next lemma makes use of the integral property (5.27) of the extended generator to establish a relationship between the conditional expectations of f and those of ζ_3 for $f \in \mathcal{D}^*$.

<u>Lemma</u> 6.4 <u>Let</u> $x(t,\eta)$ <u>satisfy</u> (6.1) - (6.3). <u>If</u> $f \in \mathcal{D}^*$, $\zeta_3(t)$ <u>is given by</u> (6.7), <u>and</u> g' <u>is a Borel measurable function which satisfies</u>

$$(6.35) \qquad E|\zeta_3(t) g'(x(t))| < \infty ,$$

<u>then</u>

$$(6.36) \qquad \frac{1}{\zeta_2(t,\tilde{\eta},w)}\left[\int_{\Omega_X} \zeta_1(t,\eta,\tilde{\eta},w)\zeta_3(t,\eta) g'(x(t,\eta)) P_X(d\eta)\right] = E^t[f(\tilde{x}(t))g'(\tilde{x}(t))](\tilde{\eta},w) \qquad \text{a.s. } P.$$

Proof: On the space $(\tilde{\Omega}_X, \tilde{\mathcal{B}}_X)$, let $\tilde{\mathcal{B}}_X^t$ be the σ-field generated by the processes $x(\tau,\tilde{\eta})$ for $0 \le \tau \le t$, and on $(W, \mathcal{B}_X)$ let $\mathcal{B}_W^t$ be generated by $w(\tau,w) = w(\tau)$ for $0 \le \tau \le t$. The σ-field $\mathcal{G}_{X,W}^t$ defined on $(\Omega, \mathcal{G})$ by

$$(6.37) \qquad \mathcal{G}_{X,W}^t = \tilde{\mathcal{B}}_X^t \times \mathcal{B}_W^t$$

is generated by $x(\tau,\eta)$ and $w(\tau,w)$ for $0 \le \tau \le t$. Since the σ-field $\mathcal{G}_Z^t$ is generated by

$$(6.38) \qquad z(\tau,\tilde{\eta},w) = \int_0^\tau x(u,\tilde{\eta})du + w(\tau)$$

for $0 \le \tau \le t$, clearly

$$(6.39) \qquad \mathcal{G}_Z^t \subset \mathcal{G}_{X,W}^t .$$

From assumption (6.35), Theorem 2.1 applies to the function

$$(6.40) \qquad g(\eta) = \zeta_3(t,\eta)g'(x(t,\eta))$$

and hence

$$(6.41) \qquad \frac{1}{\zeta_2(t)} \int_{\Omega_X} \zeta_1(t)\zeta_3(t)g'(x(t))P_X(d\eta) = E^t[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] \qquad \text{a.s. } P .$$

From the smoothing property of conditional expectations and (6.39) ([14], p. 350)

$$(6.42) \qquad E^t[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] = E^{\mathcal{G}_Z^t}[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] = E^{\mathcal{G}_Z^t}E^{\mathcal{G}_{X,W}^t}[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] \qquad \text{a.s. } P .$$

From (6.37), the fact that P is a product measure (that is, $\tilde{\eta}$ and w are independent) and $\tilde{\zeta}_3(t)g'(\tilde{x}(t))$ depends only on $\tilde{\eta}$, it can easily be shown that

$$(6.43) \qquad E^{\mathcal{G}_{X,W}^t}[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] = E^{\tilde{\mathcal{B}}_X^t}[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] \qquad \text{a.s. } P .$$

From the definition of $\tilde{\mathcal{B}}_X^t$, $g'(\tilde{x}(t))$ is measurable with respect to $\tilde{\mathcal{B}}_X^t$. Thus

$$E^{\tilde{\mathcal{B}}_X^t}[\tilde{\zeta}_3(t)g'(\tilde{x}(t))] = g'(\tilde{x}(t))E^{\tilde{\mathcal{B}}_X^t}[\tilde{\zeta}_3(t))] \qquad \text{a.s. } P\,. \tag{6.44}$$

From the definition of ζ_3 and the finite expectations of the two terms in ζ_3 assured by (5.25) and (5.26) we have

$$\begin{aligned} E^{\tilde{\mathcal{B}}_X^t}[\tilde{\zeta}_3(t)] &= E^{\tilde{\mathcal{B}}_X^t}[f(\tilde{x}(T)) - \int_t^T (G_u^* f)(\tilde{x}(u))du] \\ &= E^{\tilde{\mathcal{B}}_X^t}[f(\tilde{x}(T))] - E^{\tilde{\mathcal{B}}_X^t}[\int_t^T (G_u^* f)(\tilde{x}(u))du] \qquad \text{a.s. } P\,. \end{aligned} \tag{6.45}$$

Since the random variable

$$\int_t^T (G_u^* f)(\tilde{x}(u))du \tag{6.46}$$

clearly is measurable with respect the σ-field generated by $\tilde{x}(u)$, $t \le u \le T$, from the Markov property of $x(\tau,\eta)$ it follows that

$$E^{\tilde{\mathcal{B}}_X^t}[\int_t^T (G_u^* f)(\tilde{x}(u))du] = E[\int_t^T (G_u^* f)(\tilde{x}(u))du \,|\, \tilde{x}(t)]\,. \tag{6.47}$$

It will be verified directly that

$$E[\int_t^T (G_u^* f)(\tilde{x}(u))du \,|\, \tilde{x}(t)] = \int_t^T E[(G_u^* f)(\tilde{x}(u)) \,|\, \tilde{x}(t)]du\,. \tag{6.48}$$

From (5.1) and (5.2)

$$\int_t^T E[(G_u^* f)(\tilde{x}(u)) \,|\, \tilde{x}(t)]du = \int_t^T \int_{-\infty}^{\infty} (G_u^* f)(y)P(t,x(t,\tilde{\eta});u,dy)du\,. \tag{6.49}$$

From the assumption (6.2) of joint measurability of $P(t,x;u,A)$ and the measurability of $x(t,\tilde{\eta})$, it is clear that

$$\int_{-\infty}^{\infty} (G_u^* f)(y)P(t,x(t,\tilde{\eta});u,dy) \tag{6.50}$$

is measurable in u. From (5.26), the expression (6.49) is integrable. Thus the integral on the right side of (6.49) is well defined and is measurable with respect to the process $\tilde{x}(t,\eta)$. If B is a measurable set with respect to the σ-field generated by $\tilde{x}(t)$, by the

Fubini Theorem and the properties of conditional expectations,

$$\int_B [\int_t^T E[(G_u^* f)(\tilde{x}(u)) | \tilde{x}(t)] du] P_X(d\tilde{\eta})$$

$$= \int_t^T [\int_B E[(G_u^* f)(\tilde{x}(u)) | \tilde{x}(t)] P_X(d\tilde{\eta})] du$$

$$(6.51) \qquad = \int_t^T [\int_B ((G_u^* f)(\tilde{x}(u,\eta)) P_X(d\tilde{\eta})] du$$

$$= \int_B [\int_t^T ((G_u^* f)(\tilde{x}(u,\tilde{\eta})) du] P_X(d\tilde{\eta}) \ .$$

Thus (6.48) has been established. From (6.45), the Markov property, (6.48), the definition of the operators P_s^t in (5.1) and (5.2) and assumption (5.27) for $f \in \mathcal{D}^*$

$$E^{\tilde{\mathcal{B}}_X^t}[\tilde{\zeta}_3(t)] = E[f(\tilde{x}(T)) | \tilde{x}(t)] - \int_t^T E[((G_u^* f)(\tilde{x}(u)) | \tilde{x}(t)] du$$

$$(6.52) \qquad = (P_t^T f)(\tilde{x}(t)) - \int_t^T (P_t^u G_u^* f)(\tilde{x}(t)) du$$

$$= (P_t^T f)(\tilde{x}(t)) - [P_t^T f(\tilde{x}(t)) - f(\tilde{x}(t))] = f(\tilde{x}(t)) \qquad \text{a.s. } P.$$

The result (6.36) of the lemma then follows from (6.41), (6.42), (6.43), (6.44) and (6.52).

Next we obtain a stochastic differential for the quotient $\zeta_5(t)/\zeta_2(t)$. A difficulty arises here because the function $\Gamma(x) = x_5/x_2$ which would be required in the direct application of Lemma 3.7 is not continuous in the range $-\infty < x_5 < \infty$, $-\infty < x_2 < \infty$ as is required by the lemma. This difficulty is overcome in Lemma 6.5 and Theorem 6.1.

Lemma 6.5 *Let* $x(t)$ *satisfy* (6.1) - (6.5) *and* $f \in \mathcal{D}^*$ *be such that*

$$(6.53) \qquad \int_0^T E[\zeta_3^4(t) x^4(t)] dt < \infty$$

where ζ_3 *is given by* (6.7). *Then for* $\epsilon > 0$, *the process* $\zeta_\epsilon(t)$

<u>defined by</u>

$$\zeta_3(t) = \zeta_5(t) \cdot [\zeta_2^2(t) + \epsilon]^{-\frac{1}{2}} \tag{6.54}$$

<u>has a stochastic differential</u>

$$d\zeta_\epsilon(t) = a_\epsilon(t)dt + b_\epsilon(t)dw(t) \qquad 0 \le t \le T \tag{6.55}$$

<u>where</u>

$$\begin{aligned} a_\epsilon(t) = {} & \zeta_2(t)[\zeta_2^2(t)+\epsilon]^{-\frac{1}{2}} E^t[(G_t f)(\tilde{x}(t))] \\ & + \zeta_2(t)\tilde{x}(t)[\zeta_2^2(t)+\epsilon]^{-\frac{1}{2}} E^t[\tilde{x}(t)f(\tilde{x}(t))] \\ & - \zeta_2^2(t)[\zeta_2^2(t)+\epsilon]^{-3/2} \zeta_5(t)\tilde{x}(t)E^t[\tilde{x}(t)] \\ & - \zeta_2^3(t)[\zeta_2^2(t)+\epsilon]^{-3/2} E^t[\tilde{x}(t)]E^t[\tilde{x}(t)f(\tilde{x}(t))] \\ & + \tfrac{1}{2}\{3\zeta_2^2(t)\zeta_5(t)[\zeta_2^2(t)+\epsilon]^{-5/2} \\ & - \zeta_5(t)[\zeta_2^2(t)+\epsilon]^{-3/2}\}\zeta_2^2(t)[E^t(\tilde{x}(t))]^2 \end{aligned} \tag{6.56}$$

<u>and</u>

$$\begin{aligned} b_\epsilon(t) = {} & \zeta_2(t)[\zeta_2^2(t)+\epsilon]^{-\frac{1}{2}} E^t[\tilde{x}(t)f(\tilde{x}(t))] \\ & - \zeta_2^2(t)\zeta_5(t)[\zeta_2^2(t)+\epsilon]^{-3/2} E^t[\tilde{x}(t)] \ . \end{aligned} \tag{6.57}$$

Proof: This follows from a direct application of Lemma 3.7 to the function

$$\Gamma_\epsilon(x) = x_5(x_2^2+\epsilon)^{-\frac{1}{2}} \quad \text{where} \quad n = 2 \quad \text{and} \quad x = (x_5,x_2) \ . \tag{6.58}$$

This function has the required continuous derivatives

$$\begin{aligned} & \frac{\partial\Gamma_\epsilon(x)}{\partial x_5} = (x_2^2+\epsilon)^{-\frac{1}{2}} \ , \qquad \frac{\partial\Gamma_\epsilon(x)}{\partial x_2} = - x_5 x_2(x_2^2+\epsilon)^{-3/2} \\ & \frac{\partial^2\Gamma_\epsilon(x)}{\partial x_5^2} = 0 \ , \qquad \frac{\partial^2\Gamma_\epsilon(x)}{\partial x_5 \partial x_2} = - x_2(x_2^2+\epsilon)^{-3/2} \quad \text{and} \\ & \frac{\partial^2\Gamma_\epsilon(x)}{\partial x_2^2} = 3x_2^2 x_5(x_2^2+\epsilon)^{-5/2} - x_5(x_2^2+\epsilon)^{-3/2} \ . \end{aligned} \tag{6.59}$$

From Lemma 6.3, ζ_5 has a differential

(6.60) $$d\zeta_5(t) = a_5(t)dt + b_5(t)dw(t) \qquad 0 \le t \le T$$

where

(6.61) $$a_5(t) = \int_{\Omega_X}[\zeta_1(t)((G_t^*f)(x(t))]P_X(d\eta) + \int_{\Omega_X}[\zeta_1(t)\zeta_3(t)x(t)]P_X(d\eta)\tilde{x}(t)$$

and

(6.62) $$b_5(t) = \int_{\Omega_X}[\zeta_1(t)\zeta_3(t)x(t)]P_X(d\eta) \quad .$$

From Theorem 2.1 for

(6.63) $$g = G_t^*f(x(t,\eta)) \quad ,$$

(6.64) $$\int_{\Omega_X}\zeta_1(t)((G_t^*f)(x(t))P_X(d\eta) = \zeta_2(t)E^t[((G_t^*f)(\tilde{x}(t))] \ .$$

From assumption (6.53), it follows that

(6.65) $$E[|\zeta_2(t)x(t)|] < \infty \qquad \text{a.e. } \mu_{[0,T]}$$

and hence that Lemma 6.4 applies for all t satisfying (6.65))

and with

(6.66) $$g'(x) = x \ .$$

Thus the coefficients a_5 and b_5 may be written

(6.67) $$a_5(t) = \zeta_2(t)E^t[((G_t^*f)(\tilde{x}(t))] + \tilde{x}(t)\zeta_2(t)E^t[\tilde{x}(t)f(\tilde{x}(t))]$$
$$\text{a.e. } \mu_{[0,T]} \times P$$

and

(6.68) $$b_5(t) = \zeta_2(t)E^t[\tilde{x}(t)f(\tilde{x}(t))] \qquad \text{a.e. } \mu_{[0,T]} \times P.$$

Also from assumptions (6.1), (6.4) and (6.5) Lemma 4.6 applies and ζ_2 has the differential (4.64)

(6.69) $$d\zeta_2(t) = a_2(t)dt + b_2(t)dw(t) \qquad 0 \le t \le T \ .$$

From (6.4), $E[|x(t)|] < \infty$ a.e. and hence Theorem 2.1 applies with $g(\eta) = x(t,\eta)$ and t for which $x(t)$ is absolutely integrable.

Thus from Theorem 2.1 the coefficients in (4.64) may be written

$$a_2(t) = \zeta_2(t)\tilde{x}(t)E^t[\tilde{x}(t)] \qquad \text{a.e. } \mu_{[0,T]} \times P \tag{6.70}$$

and

$$b_2(t) = \zeta_2(t)E^t[\tilde{x}(t)] \qquad \text{a.e. } \mu_{[0,T]} \times P . \tag{6.71}$$

<u>Theorem</u> 6.1 <u>Let</u> $x(t,\eta)$ <u>satisfy</u> (6.1) - (6.5) <u>and</u> <u>let</u> $f \in \mathcal{D}^*$ <u>satisfy</u>

$$\int_0^T E[f^4(x(t)]dt < \infty \tag{6.72}$$

<u>and</u>

$$\int_0^T E\{x(t)[f(x(T) - \int_t^T ((G_u^* f)(x(u))du]\}^4 dt < \infty \tag{6.73}$$

<u>then</u> <u>the</u> <u>process</u> $E^t[f(x(t))]$ <u>on</u> $(\Omega,\mathcal{G},P)$ <u>has</u> <u>a</u> <u>stochastic</u> <u>differential</u>

$$\begin{aligned} dE^t[f(x(t))] &= E^t[(G_t^* f)(x(t))]dt \\ &+ \{E^t[x(t)f(x(t))] - E^t[f(x(t))]E^t[x(t)]\}\{dz(t) - E^t[x(t)]dt\} \end{aligned} \tag{6.74}$$

$$0 \le t \le T .$$

In (6.74) of the theorem $x(t)$ is defined on $\tilde{\Omega}_X$. However, the notation $\tilde{x}$ for x has been omitted in the statement of the theorem since the domain is clear from the fact that the stochastic differential is defined on $\Omega = \tilde{\Omega}_X \times W$.

Proof: From Lemma 6.5, for $0 \le s < t \le T$ and $\epsilon > 0$

$$\zeta_5(t)[\zeta_2^2(t)+\epsilon]^{-\frac{1}{2}} - \zeta_5(s)[\zeta_2^2(s)+\epsilon]^{-\frac{1}{2}} = \int_s^t a_\epsilon(u)du + \int_s^t b_\epsilon(u)du, \quad \text{a.s. } P \tag{6.75}$$

where a_ϵ and b_ϵ are given by (6.56) and (6.57). Since $0 < \zeta_2(u) < \infty$ a.s. P, for $\epsilon_n \to 0$ $(n \to \infty)$ from routine computations it follows that

$$(6.76)\qquad \zeta_5(t)[\zeta_2^2(t)+\epsilon_n]^{-\frac{1}{2}} - \zeta_5(s)[\zeta_2^2(s)+\epsilon_n]^{-\frac{1}{2}} \to \frac{\zeta_5(t)}{\zeta_2(t)} - \frac{\zeta_5(s)}{\zeta_2(s)} \quad \text{a.s. } P,$$

for $s \le u \le t$,

$$(6.77)\qquad \begin{aligned} a_{\epsilon_n}(u) \to\ & E^u[(G_u f)(\tilde{x}(u))] \\ & + \{E^u[\tilde{x}(u)f(\tilde{x}(u))] - \frac{\zeta_5(u)}{\zeta_2(u)} E^u[\tilde{x}(u)]\}\{\tilde{x}(u)-E(\tilde{x}(u))\} \end{aligned} \quad \text{a.s. } P,$$

and

$$(6.78)\qquad b_{\epsilon_n}(u) \to E^u[\tilde{x}(u)f(\tilde{x}(u))] - \frac{\zeta_5(u)}{\zeta_2(u)} E^u[\tilde{x}(u)] \qquad \text{a.s. } P\ .$$

from (6.26) (6.35) is true with $g'(x) \equiv 1$ and Lemma 6.4 holds for all u and $g' \equiv 1$. Hence from (6.36), for $0 \le u \le T$

$$(6.79)\qquad \frac{\zeta_5(u)}{\zeta_2(u)} = E^u[f(\tilde{x}(u))] \qquad \text{a.s. } P\ .$$

The following bounds are easily obtained. For $0 \le u \le T$ and $n \ge 1$

$$(6.80)\qquad |a_{\epsilon_n}(u)| \le \bar{a}(u) \quad \text{and} \quad |b_{\epsilon_n}(u)| \le \bar{b}(u)$$

where

$$(6.81)\qquad \begin{aligned} \bar{a}(u) = & |E^u[(G_u f)\tilde{x}(u)]| \\ & + |\tilde{x}(u)||E^u[\tilde{x}(u)f(\tilde{x}(u))]| + E^u[|f(\tilde{x}(u))|]|\tilde{x}(u)||E^u[\tilde{x}(u)]| \\ & + |E^u[\tilde{x}(u)]||E^u[\tilde{x}(u)f(\tilde{x}(u))]| + 2E^u[|f(\tilde{x}(u))|]\{E^u[\tilde{x}(u)]\}^2 \end{aligned} \quad \text{a.s. } P\ ,$$

$$\bar{b}(u) = |E^u[\tilde{x}(u)f(\tilde{x}(u))]| + E^u[|f(\tilde{x}(u))|]|E^u(\tilde{x}(u))| \qquad \text{a.s. } P\ .$$

It will be shown next $\bar{a} \in \mathcal{M}_1$ and $\bar{b} \in \mathcal{M}_2$. The measurability properties (3.5) and (3.6) are easily established.

Since from (6.73)

$$(6.82)\qquad E[|\tilde{\zeta}_3(t)\tilde{x}(t)|] < \infty \qquad \text{a.s. } \mu_{[0,T]}$$

for t for which (6.82) holds Lemma 6.4 applies with $g'(x) = x$

and Theorem 2.1 applies for $g(\eta) = \zeta_3(t,\eta)x(t,\eta)$. From these, we conclude that

$$E^t[\tilde{x}(t)f(\tilde{x}(t))] = E^t[\tilde{x}(t)\tilde{\zeta}_3(t)] \qquad \text{a.e. } \mu_{[0,T]} \times P. \tag{6.83}$$

From (6.83), the properties of conditional expectations and assumptions (6.4), (6.72) and (5.26) it is easily shown that

$$E[\int_0^T |\bar{a}(u)|du] < \infty \quad \text{and} \quad E[\int_0^T [\bar{b}(u)]^2 du] < \infty . \tag{6.84}$$

Thus Lemma 3.4 applies to the sequences (6.77) and (6.78), and we conclude that

$$\begin{aligned}\int_s^t a_{\epsilon_n}(u)du \xrightarrow{P} \int_s^t \{&E^u[(G_u^* f)(\tilde{x}(u))] \\ &+ \left[E^u[\tilde{x}(u)f(\tilde{x}(u))] - E^u[f(\tilde{x}(u))]E^u[\tilde{x}(u)]\right]\left[\tilde{x}(u)-E^u[\tilde{x}(u)]\right]\}du \\ &\text{a.s. } P\end{aligned} \tag{6.85}$$

and

$$\int_s^t b_{\epsilon_n}(u)du \xrightarrow{P} \int_s^t \{E^u[\tilde{x}(u)f(\tilde{x}(u))] - E^u[f(\tilde{x}(u))]E^u[\tilde{x}(u)]\}dw(u) \quad \text{a.s. } P. \tag{6.86}$$

The result (6.73) of the theorem with $dz(t)$ replaced by $\tilde{x}(t)dt+dw(t)$ then follows from (6.75), (6.76), (6.79), (6.85) and (6.86). The differential

$$N(f,t)dz(t) = N(f,t)\tilde{x}(t) + N(f,t)dw(t) \tag{6.87}$$

is defined by (3.118) and (3.119) where for convenience we let

$$N(f,t) = E^t[\tilde{x}(t)f(\tilde{x}(t))] - E^t[f(\tilde{x}(t))]E^t[\tilde{x}(t)] . \tag{6.88}$$

From (1.1) $z(t)$ has a differential on $(\Omega, \mathcal{B}, P)$

$$dz(t) = \tilde{x}(t)dt + dw(t) \qquad 0 \le t \le T. \tag{6.89}$$

In order to justify (6.87) we need only verify that

$$N(f,t)\tilde{x}(t) \in \mathcal{M}_1 \quad \text{and} \quad N(f,t) \in \mathcal{M}_2 . \tag{6.90}$$

The measurability requirements (3.5) and (3.6) follow from

$$(6.91)\qquad \begin{aligned} b_{\epsilon_n}(u) &\to N(f,u) && \text{a.s. } P \\ b_{\epsilon_n}(u)\tilde{x} &\to N(f,u)\tilde{x}(u) && \text{a.s. } P \end{aligned}$$

where $b_{\epsilon_n}(u)$ and $b_{\epsilon_n}(u)\tilde{x}(u)$ satisfy (3.5) and (3.6). The integrability requirements follow from (6.80), (6.84) and (6.4).

It should be noticed that the intepretation of the integral of (6.87) as a limit of Ito sums (3.121) as given in Lemma 3.8 is not valid here since we have not shown that $N(f,t)$ has a stochastic differential. Under the slightly stronger assumptions of Theorem 7.1, $N(f,t)$ will be shown to have a differential and hence under the assumptions of Theorem 7.1, Lemma 3.8 will apply.

As a corollary we obtain

Theorem 6.2 *If $x(t)$ satisfies (6.1) - (6.5) and $f \in \mathcal{D}$ (the domain of the strong generator) then the process $E^t[f(x(t))]$ on $(\Omega, \mathcal{G}, P)$ has a stochastic differential*

$$dE^t[f(x(t))] = E^t[(G_t f)(x(t))]dt$$

$$(6.92)\qquad + \{E^t[x(t)f(x(t))] - E^t[f(x(t))]E^t[x(t)]\}\{dz(t) - E^t[x(t)]dt\}$$

$$0 \le t \le T .$$

Proof: For $f \in \mathcal{D}$, $f(x)$ and $(G_t f)(x)$ are bounded and hence (6.72) holds and (6.73) follows from (6.4). The result then follows from Theorem 6.1. The notation $\tilde{x}$ is again omitted in equation (6.92).

7. A Stochastic Differential Equation of the Fisk-Stratonovich Type.

As mentioned in the Introduction the interpretation of the stochastic differential equation obtained in [18] and [20] cannot be given in terms of the stochastic integrals of Ito. We propose in this section to prove a precise analogue of Theorem 6.1 yielding a stochastic differential equation involving integrals of the Fisk-Stratonovich type. Before proceeding to our theorem we recapitulate briefly the definition of the Fisk integral and state a specific result connecting the Fisk and Ito integrals that will prove useful to us. The stochastic integral of Fisk is defined for processes $M(t)$, $0 \le t \le T$, on a probability space $(\Omega, \mathcal{G}, P)$, which have the following properties:

$$(7.1) \qquad M(t) \text{ is } \widetilde{\mathcal{F}}_t\text{-measurable} \qquad \text{for } 0 \le t \le T$$

where the complete σ-fields $\widetilde{\mathcal{F}}_t$ are monotone increasing (3.2); for $0 \le t \le T$

$$(7.2) \qquad M(t) = M^{(1)}(t) + M^{(2)}(t) \qquad \text{a.s. } P$$

where $M^{(1)}(t)$ and $M^{(2)}(t)$ have continuous sample functions, $M^{(1)}(t)$ is a martingale and

$$(7.3) \qquad \underset{0 \le t \le T}{\text{variation}}\, M^{(2)}(t) = V$$

where

$$(7.4) \qquad E(V) < \infty .$$

It may be noticed that the assumption of a.s. continuous sample functions implies that $M(t)$ is a jointly measurable process for the completed σ-field $\overline{\mathcal{G}}$.

If $M(t)$ and $N(t)$ are quasi martingales with respect to the same σ-fields $\widetilde{\mathcal{F}}_t$, then the Fisk (F) integral is defined by

(7.5) $$(F)\int_0^T N(u)dM(u) = \underset{\Pi_n}{\text{plim}}\tfrac{1}{2}\Sigma[N(t_j^n)+N(t_{j+1}^n)][M(t_{j+1}^n)-M(t_j^n)]$$

where the right hand side limit exists in probability as $\max_j(t_{j+1}^n - t_j^n) \to 0 \quad (n \to \infty)$ for the subdivision $\Pi_n = \{t_j^n\}$ of $[0,T]$. It is clear how (7.5) yields the definition of the F-integral over any sub-interval $[s,t]$ of $[0,T]$. Fisk [5] has shown that

(7.6) $$\begin{aligned}(F)\int_0^T N(u)dM(u) &= (I)\int_0^T N(u)dM(u) \\ &+ \tfrac{1}{2}\underset{\Pi_n}{\text{plim}}\ \Sigma[N(t_{j+1}^n)-N(t_j^n)][M(t_{j+1}^n)-M(t_j^n)] .\end{aligned}$$

The Ito integral, on the right hand side in (7.6) is by definition the limit in probability of the Ito sums (3.121). It may be noticed that for integrals defined in this generality there is no reasonable interpretation for the equation (3.118). However, we shall specialize immediately to the case in which the definition (3.118) does apply. We will be concerned here only with this case.

Suppose that $M(t)$ and $N(t)$ possess the (Ito) stochastic differentials

(7.7) $$\begin{aligned} dM(t) &= a_1(t)dt + b_1(t)dw(t), \\ dN(t) &= a_2(t)dt + b_2(t)dw(t) \quad . \end{aligned}$$

Here $w(t)$ is a standard Wiener process, and we assume that a separable version of the integral $\int_0^t b_i(u)dw(u)$ is chosen and that

(7.8) $$\int_0^T E|a_i(u)|du \quad \text{and} \quad \int_0^T Eb_i^2(u)du < \infty \qquad (i = 1,2).$$

Then it is easily seen that $M(t)$ and $N(t)$ are quasi martingales with respect to the σ-fields $\mathfrak{F}_t$ which occur in the definition of the above stochastic integrals. In this case the relation (7.6) between the F- and I-integrals takes a particularly simple form which we state as a lemma.

<u>Lemma</u> 7.1 <u>If</u> $M(t)$ <u>and</u> $N(t)$ <u>are quasi-martingales possessing stochastic differentials</u> (7.7) <u>which satisfy</u> (7.8)

$$(7.9) \qquad (F)\int_s^t N(u)dM(u) = (I)\int_s^t N(u)dM(u) + \tfrac{1}{2}\int_s^t b_1(u)b_2(u)du \,.$$

This result follows easily from the equation (7.6) established by Fisk. Here the conditions (3.118) and (3.120) of Lemma 3.8 hold so that the Ito integral on the right of (7.9) satisfies both (3.118) and (3.121). It remains only to identify the second term on the right of (7.9). The details of this will be left to the reader.

If we take $N(t) = \Phi[t,M(t)]$ where $\Phi(t,x)$ is assumed to be continuous in (t,x) on $[0,T] \times (-\infty,\infty)$ and $\frac{\partial\Phi}{\partial x}(t,x)$ is also assumed to exist and be continuous, then (7.4) gives us the integral of Stratonovich,

$$(7.10) \qquad (S)\int_s^t \Phi[u,M(u)]dM(u) = (I)\int_s^t \Phi[u,M(u)]dM(u) + \tfrac{1}{2}\int_s^t \frac{\partial\Phi}{\partial x}[u,M(u)] \cdot b_1(u)du \,.$$

It can be seen that the above relation due to Stratonovich by means of which he defines the symmetrized integral on the left is too specialized to help us convert the Ito integral in (6.74) into the corresponding Fisk (or symmetrized) integral. Lemma 7.1, given above, deals with the relation between Fisk and Ito integrals at the level of generality required for our purpose. With its help and Theorem 6.1 we prove a result which provides conditions under which $\mathcal{E}^t[f(\underset{\sim}{x}(t))]$ satisfies a stochastic differential of the Fisk-Stratonovich type. To the best of our knowledge such a derivation is, so far, not available in the literature.

Since we will be concerned now only with the space $\Omega = \tilde{\Omega}_X \times W$ we will for convenience designate elements of $\tilde{\Omega}_X$ by η.

<u>Theorem</u> 7.1 <u>Let</u> $x(t,\eta)$ <u>be a process which satisfies</u> (6.1) - (6.3), (6.5) <u>and let</u> $f(x)$ <u>be a real valued function for which</u>

(7.11) $$x \in \mathcal{D}^*, \quad f(x) \in \mathcal{D}^*, \quad \text{and} \quad xf(x) \in \mathcal{D}^* ;$$

(7.12) $$E[x(T)]^8 < \infty, \quad E[f(x(T))]^8 < \infty, \quad \text{and} \quad E[x(T)f(x(T))]^8 < \infty ;$$

(7.13) $$\int_0^T E[x(t)]^8 dt < \infty, \quad \int_0^T E[f(x(t))]^4 dt < \infty, \quad \text{and} \int_0^T E[x(t)f(x(t))]^4 dt < \infty;$$

and

(7.14) $$\int_0^T E[(G_t^* x)(x(t))]^8 dt < \infty, \quad \int_0^T E[(G_t^* f)(x(t))]^8 dt < \infty, \quad \text{and}$$
$$\int_0^T [(G_t^* xf(x))(x(t))]^8 dt < \infty .$$

<u>Then</u> $E^t[f(x(t))]$ <u>satisfies the stochastic differential equation of Ito type</u> (6.74) <u>of Theorem</u> (6.1); <u>it also satisfies the stochastic equation of Fisk type, for</u> $0 \le s < t \le T$

(7.15)
$$E^t[f(x(t))] - E^s[f(x(s))]$$
$$= (F)\int_s^t \{E^u[x(u)f(x(u))] - E^u[f(x(u))]E^u[x(u)]\}dz(u)$$
$$+ \int_s^t E^u[(G_u^* f)(x(u))]du - \tfrac{1}{2}\int_s^t \{E^u[x^2(u)f(x(u))] - E^u[f(x(u))]E^u[x^2(u)]\}du$$
a.s. P .

Proof: We will first show that Theorem 6.1 applies for the three functions x, $f(x)$, and $xf(x)$. By assumption (7.11) they all belong to $\mathcal{D}^*$, (7.13) implies (6.4) and (6.72) for the three functions. It remains to verify (6.73). This follows in a straightforward manner from the assumptions (7.12) - (7.14) and by repeated applications of the Schwarz and Holder inequalities and the elementary inequality $(|a|+|b|)^p \le 2^{p-1}(|a|^p+|b|^p)$ for $p > 1$. Thus Theorem 6.1 applies for the function f. In the proof of Theorem 6.1, it was shown that

$$N(f,t)dz(t) = N(f,t)x(t)dt + N(f,t)dw(t) \qquad 0 \le t \le T \tag{7.16}$$

where $N(f,t)$ is defined by (6.88). Using this in (6.74), we may write the differential of Ito type

$$\begin{aligned} dE^t[f(x(t))] &= E^t[(G_t^* f)(x(t))]dt \\ &\quad + N(f,t)[\{x(t) - E^t[x(t)]\}dt + dw(t)] \\ &\qquad\qquad 0 \le t \le T . \end{aligned} \tag{7.17}$$

Now substituting x for $f(x)$ in (6.74) we obtain the differential for $E^t[x(t)]$

$$dE^t[x(t)] = \alpha_1(t)dt + \beta_1(t)dw(t) \qquad 0 \le t \le T \tag{7.18}$$

where

$$\alpha_1(t) = E^t[(G_t^* x)(x(t))] + x(t) - E^t[x(t)] \tag{7.19}$$

and

$$\beta_1(t) = E^t([x(t)]^2) - [E^t(x(t))]^2 . \tag{7.20}$$

Applying Ito's lemma on stochastic differentials (Lemma 3.7) to the process $E^t[f(x(t))]E^t[x(t)]$ we obtain from (7.17) and (7.18) the following formula.

$$d\{E^t[f(x(t))]\, E^t[x(t)]\} = \alpha_2(t)dt + \beta_2(t)dw(t) , \tag{7.21}$$

where

$$\begin{aligned} \alpha_2(t) &= \alpha_1(t)E^t[f(x(t))] + \{N(f,t)[x(t)-E^t(x(t))] \\ &\quad + E^t[(G_t^* f)(x(t))]\} \cdot E^t(x(t)) + \beta_1(t)N(f,t) \end{aligned} \tag{7.22}$$

and

$$\beta_2(t) = E^t[f(x(t))]\beta_1(t) + E^t(x(t))N(f,t) . \tag{7.23}$$

We now repeat the above procedure, this time replacing f by xf in (6.74) to obtain

$$dE^t[x(t)f(x(t))] = \alpha_3(t)dt + \beta_3(t)dw(t) \qquad 0 \le t \le T \tag{7.24}$$

where

$$\begin{aligned}\alpha_3(t) = {} & E^t[(G_t^* xf)(x(t))] + [E^t\{x^2(t)f(x(t))\} \\ & - E^t(x(t))E^t\{x(t)f(x(t))\}] \cdot [x(t) - E^t(x(t))]\end{aligned} \tag{7.25}$$

and

$$\beta_3(t) = E^t[x^2(t)f(x(t))] - E^t[x(t)]\cdot E^t[x(t)f(x(t))] . \tag{7.26}$$

Hence from (7.21) and (7.24) it follows that the process $N(f,t)$ defined by (6.88) ($f \in \mathcal{D}^*$) has the stochastic differential

$$dN(f,t) = \alpha(t)dt + \beta(t)dw(t) \qquad 0 \le t \le T \tag{7.27}$$

where

$$\alpha(t) = \alpha_3(t) - \alpha_2(t) , \text{ and } \beta(t) = \beta_3(t) - \beta_2(t) . \tag{7.28}$$

From (6.88), (7.20), (7.23) and (7.26)

$$\begin{aligned}\beta(t) = {} & E^t[x^2(t)f(x(t))] - E^t[x(t)] \cdot E^t[x(t)f(x(t))] \\ & - \{E^t[f(x(t))][E^t(x^2(t)) - (E^t(x(t)))^2] \\ & + E^t(x(t))\cdot[E^t(x(t)f(x(t))) - E^t(f(x(t)))\cdot E^t(x(t))]\} \\ = {} & E^t[x^2(t)f(x(t))] - E^t[x^2(t)]\cdot E^t[f(x(t))] \\ & + 2\{[E^t(x(t))]^2\cdot E^t[f(x(t))] - E^t(x(t))\cdot E^t[x(t)f(x(t))]\} .\end{aligned} \tag{7.29}$$

By repeated use of Schwarz inequality and inequalities for conditional expectations it can easily be shown from the assumptions (7.13) and

(7.14) that

(7.30) $$\int_0^T E|\alpha(t)|dt < \infty \quad \text{and} \quad \int_0^T E[\beta(t)]^2 dt < \infty .$$

From (1.1) $z(t)$ has an Ito differential

(7.31) $$dz(t) = x(t)dt + dw(t) \qquad 0 \le t \le T .$$

Thus from (7.27), (7.30) and (7.13) $N(f,t)$ and $z(t)$ satisfy the conditions of Lemma 7.1. From this lemma we obtain

$$(F)\int_s^t N(f,u)dz(u) = (I)\int_s^t N(f,u)dz(u)$$

(7.32) $$+ \tfrac{1}{2}\int_s^t [E^u(x^2(u)f(x(u))) - E^u(x^2(u))E^u(f(x(u)))]du$$

$$+ \int_s^t [\{E^u(x(u))\}^2\{E^u(f(x(u)))\} - \{E^u(x(u))E^u(x(u)f(x(u)))\}]du.$$

Finally substituting the Fisk integral for the Ito integral $\int_s^t N(f,u)dz(u)$ from (7.32) in the basic equation (6.74) of Theorem 6.1 we obtain the stochastic equation (7.15) of the theorem.

It may be noted here that under the assumption of Theorem 7.1 the Ito integral dz obtained in equation (6.74) may be interpreted as a limit of Ito sums (3.121) since from (7.27) Lemma 3.8 applies.

Some comment concerning the assumptions in Theorem 7.1 is perhaps in order. Certainly assumptions (6.1) - (6.3) and (7.11) are essential if our particular line of argument is to succeed. Assumption (6.5) is discussed at the end of Section 8. The order of the moment assumptions (7.12) - (7.14) is somewhat arbitrary. The moments four and eight can undoubtedly be reduced by adjusting the coefficient 16 in the condition (6.5). No attempt has been made to obtain the "best" moment conditions since for the examples in which we are primarily interested moments of all orders are finite and integrable (see Theorems 8.1, 8.2 and Lemma 8.7).

8. Application to System Processes Satisfying Stochastic Differential Equations. We shall show in this section that among the system processes to which Theorems 6.1 and 7.1 can be applied, are a rather large class of diffusion processes. First, we prove several preparatory lemmas which are essentially stages in the proof of the result we seek. The first lemma is concerned with condition (6.5) which underlies both of our main theorems. Restrictive as it looks, it is a weaker condition than requiring $E\left[e^{16\int_0^T x^2(t)dt}\right]$ to be finite. For a trivial example which brings out the difference between the two conditions, consider $x(t) = x_0$ (identically in t) where x_0 is a random variable with a standard Gaussian distribution. Then $E\left(e^{\lambda x_0^2}\right)$ is finite if $0 < \lambda < \frac{1}{2}$ and $= \infty$ for $\lambda > \frac{1}{2}$. It is easy to see that (6.5) is satisfied whatever the value of T, whereas $E\left[e^{16\int_0^T x^2(t)dt}\right]$ is finite if and only if T, which is the length of the interval of observation is less than 1/32. In the following lemma we exhibit a class of diffusion processes $x(t)$ for which (6.5) holds.

Let $x(t)$ $(0 \le t \le T)$ be the solution of the Ito stochastic differential equation

$$dx(t) = m[t,x(t)]dt + dB(t) \qquad 0 \le t \le T \tag{8.1}$$

or, equivalently of the equation

$$x(t) = x(s) + \int_s^t m[u,x(u)]du + B(t)-B(s) \qquad (0 \le s < t \le T), \tag{8.2}$$

where $B(t)$ is a standard Wiener process, $m(u,\xi)$ $(-\infty < \xi < \infty)$ satisfies the Doob-Ito conditions

(8.3)
$$|m(u,\xi)| \le K(1+\xi^2)^{\frac{1}{2}},$$
$$|m(u,\xi_2)-m(u,\xi_1)| \le K|\xi_1-\xi_2|$$

uniformly in t for some constant K, and $x(0)$ is a random variable independent of $\{B(t), t \geq 0\}$. Under these conditions it is well known ([2], Chapter VI) that $x(t)$ is a uniquely determined Markov process almost all of whose sample functions are continuous. It should be noted that the process $x(t)$ depends on the choice of $x(0)$ which is arbitrary.

Lemma 8.1 *Let* $x(t)$ $(0 \leq t \leq T)$ *be the solution of* (8.1) *where, in addition to* (8.3) *it is assumed that the initial random variable* $x(0)$ *satisfies*

$$(8.4) \qquad E\left[e^{cx^2(0)}\right] < \infty \qquad \text{for some } c > 0 .$$

Then condition (6.5) *of Theorem* 6.1 *holds.*

For the proof of the lemma we shall need the following two auxiliary results.

Lemma 8.2 *Let* $B(u)$ $(u \geq 0)$ *be a standard Wiener process. If* $\lambda > 0$ *and* $h > 0$, *then*

$$(8.5) \qquad E\left\{e^{\lambda\int_0^h B^2(u)du}\right\} < \infty \qquad \text{provided } \lambda h^2 < \frac{3}{4} .$$

Proof: The process $y(v) = \frac{1}{\sqrt{h}} B(hv)$ is clearly a standard Wiener process on $[0,1]$ and $E\left\{e^{\lambda\int_0^h B^2(u)du}\right\} = E\left\{e^{\lambda h^2\int_0^1 y^2(v)dv}\right\}$.

From the usual orthogonal (Fourier series) expansion for the Wiener process ([6], pp. 248-249) we obtain the inequality

$\int_0^1 y^2(v)dv \leq \frac{2}{3} y_0^2 + 2\int_0^1 y_1^2(v)dv$, where $y_1(v)$ $(0 \leq v \leq 1)$ is Gaussian with mean 0 and covariance $E[y_1(u)y_1(v)] = \min(u,v) - u\cdot v$; and y_0 is a standard normal random variable independent of the process $\{y_1(v)\}$.

Hence from formula (4) of [6] (p. 640)

$$E\left\{e^{\lambda\int_0^h B^2(u)du}\right\} \le E\left(e^{\frac{2h^2\lambda}{3}y_0^2}\right) \cdot E\left[e^{2h^2\lambda\int_0^1 y_1^2(v)dv}\right]$$

$$= E\left(e^{\frac{2h^2\lambda}{3}y_0^2}\right) \cdot \prod_{k=1}^{\infty}\left(1 - \frac{4h^2\lambda}{k^2\pi^2}\right)^{-\frac{1}{2}} < \infty$$

provided $h^2\lambda < \frac{3}{4}$.

Lemma 8.3 Let $B(t)$ $(0 \le t \le T)$ be as above. Then for every λ and h positive,

$$E\left\{e^{\lambda\int_t^{t+h}[B(u)-B(t)]^2du}\right\} < \infty \tag{8.6}$$

for all t such that $0 \le t < t+h \le T$, provided

$$h^2\lambda < \frac{3}{4} . \tag{8.7}$$

The proof follows at once from the preceding lemma if we note that $\int_t^{t+h}[B(u)-B(t)]^2du$ has the same distribution as $\int_0^h B^2(u)du$.

Proof of Lemma 8.1: From (8.2) writing

$$x(u)-x(t) = \int_t^u m[s,x(s)]ds + B(u)-B(t) \quad (0 \le t < u \le T),$$

applying the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ and the Schwarz inequality to the right hand side and using the first of the two conditions in (8.3) we have

$$[x(u)-x(t)]^2 \le 2K^2(u-t)\int_t^u[1+x^2(s)]ds + 2[B(u)-B(t)]^2 .$$

Letting $h > 0$ such that $\alpha = 1-2K^2h^2$ is positive, for each t with $0 \le t < t+h \le T$

$$\int_t^{t+h} x^2(u)du \le 2\int_t^{t+h}[x(u)-x(t)]^2du + 2hx^2(t)$$

$$\le 2K^2h^2\int_t^{t+h}[1+x^2(s)]ds + 4\int_t^{t+h}[B(u)-B(t)]^2du + 2hx^2(t) .$$

From this follows the inequality

$$(8.8) \qquad \int_t^{t+h} x^2(u)du \le \frac{2K^2h^3}{\alpha} + \frac{4}{\alpha}\int_t^{t+h} [B(u)-B(t)]^2du + \frac{2h}{\alpha} x^2(t) .$$

From Lemma 8.3, provided h satisfies

$$(8.9) \qquad \frac{4\lambda h^2}{\alpha} < \frac{3}{4} ,$$

it follows that

$$(8.10) \qquad E\left\{ e^{\frac{4\lambda}{\alpha}\int_t^{t+h}[B(u)-B(t)]^2du} \right\} < \infty$$

for all t such that $0 \le t < t+h \le T$.

We now proceed as in the existence proof given in [2] (pp. 279-281) making only the changes appropriate to our present purpose. Define a sequence of approximations to the solution of (8.2) as follows. Let $x_0(t) \equiv 0$ for all t and for $n > 1$ define $x_n(t)$ recursively by the relation

$$x_n(t) = x(0) + \int_0^t m[s,x_{n-1}(s)]ds + B(t) ,$$

where $x(0)$ is chosen as in the statement of Lemma 8.1. For $n \ge 1$ let

$$\Delta_n x(t) = x_n(t) - x_{n-1}(t) \quad \text{and} \quad \Delta_n m(t) = m[t,x_n(t)] - m[t,x_{n-1}(t)].$$

Clearly since $\Delta_1 x(t) = x_1(t)$, it follows from the definition of $x_1(t)$ and (8.3) that

$$[\Delta_1 x(t)]^2 \le 4K^2t + 4B^2(t) + 2x^2(0) .$$

For $n > 1$, using the second condition in (8.3) and setting $M = K^2T$ we obtain

$$[\Delta_n x(t)]^2 \le M\int_0^t [\Delta_{n-1}x(s)]^2ds .$$

From this, proceeding as in [2] (p. 280)

$$[\Delta_n x(t)]^2 \le M^{n-1}\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\,[\Delta_1 x(s)]^2 ds$$

$$\le M^{n-1}\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\,\{4K^2 s + 4B^2(s) + 2x^2(0)\}ds$$

$$\le M^{n-1}\left[\frac{4MT^{n-1}}{(n-1)!} + \frac{4T^{n-2}}{(n-2)!}\int_0^T B^2(u)du + \frac{2T^{n-1}}{(n-1)!}\,x^2(0)\right]$$

$$= \frac{4M\cdot A^{n-1}}{(n-1)!} + \frac{4M\cdot A^{n-2}}{(n-2)!}\int_0^T B^2(u)du + \frac{2A^{n-1}}{(n-1)!}\,x^2(0), \qquad \text{a.s.},$$

where we have set $A = MT$. From the above inequality and the inequality $(n > m)$

$$[x_n(t)-x_m(t)]^2 \le \left[\sum_{j=m+1}^{n} 2^{-j}\right]\left[\sum_{j=m+1}^{n} 2^{j}\,(\Delta_j x)^2\right]$$

and reasoning as in [2] (p. 281) we arrive at two conclusions. The first is that for each t, $x_n(t)$ converges to $x(t)$ with probability one. Now taking $m = 1$, we have

$$[x_n(t)-x_1(t)]^2 \le \tfrac{1}{2}\left[\sum_{j=2}^{\infty}\frac{(2A)^{j-1}}{(j-1)!} + 16M\sum_{j=2}^{\infty}\frac{(2A)^{j-2}}{(j-2)!}\int_0^T B^2(u)du + 4\sum_{j=2}^{\infty}\frac{(2A)^{j-1}}{(j-1)!}\,x^2(0)\right]$$

$$= \tfrac{1}{2}\left[c_1 + c_2\int_0^T B^2(u)du + c_3 x^2(0)\right], \quad \text{say}.$$

From the above inequality and from

$$x_n^2(t) \le 2[x_n(t)-x_1(t)]^2 + 2x_1^2(t)$$

it follows after some computations that

$$E\left[e^{\frac{2\lambda h}{\alpha}x_n^2(t)}\right] \le e^{\frac{2\lambda h c_1}{\alpha}}\cdot E\left[e^{\frac{2\lambda h c_2}{\alpha}\int_0^T B^2(u)du + \frac{2\lambda h c_3}{\alpha}x^2(0)}\cdot e^{\frac{4\lambda h}{\alpha}x_1^2(t)}\right].$$

Applying the Schwarz inequality to the expectation on the right hand side, recalling the bound for $x_1^2(t)$ obtained earlier and noting that $x(0)$ is independent of $B(u)$ $(0 \le u \le T)$ we obtain our second conclusion

$$E\left[e^{\frac{2\lambda h}{\alpha} x_n^2(t)}\right] \leq e^{\frac{2\lambda hc_1+16\lambda hM}{\alpha}} \cdot$$

(8.11)
$$\left\{E\left[e^{\frac{4\lambda hc_2}{\alpha}\int_0^T B^2(u)du}\right] \cdot E\left[e^{\frac{32\lambda h}{\alpha} B^2(t)}\right] \cdot E\left[e^{\frac{4\lambda hc_3}{\alpha} x^2(0)}\right] \cdot E\left[e^{\frac{16\lambda h}{\alpha} x^2(0)}\right]\right\}^{\frac{1}{2}} .$$

$E\left\{e^{\frac{32\lambda h}{\alpha} B^2(b)}\right\}$ is finite for all t $(0 < t \leq T)$ if $\frac{\lambda h}{\alpha} \leq \frac{1}{64T}$.

From (8.5) of Lemma 8.2 and condition (8.4) of Lemma 8.1 it follows then that the right hand side of (8.11) is finite provided λ and h satisfy the inequality

(8.12)
$$\frac{\lambda h}{\alpha} < \min\left\{\frac{3}{16c_2T^2} , \frac{1}{64T} , \frac{c}{4c_3} , \frac{c}{16}\right\} .$$

Hence making $n \to \infty$ in (8.11) it follows from Fatou's lemma that

(8.13)
$$E\left[e^{\frac{2\lambda h}{\alpha} x^2(t)}\right] < \infty$$

for all t in $[0,T]$ if $h > 0$, $\alpha > 0$ and (8.12) holds.

It is clear that given a positive λ (the value $\lambda = 16$ is of interest to us) h can be chosen in such a way that $\alpha > 0$ and the inequalities (8.9) and (8.12) are satisfied. Any such value of h, say Δ, depends only on λ, T and K and not on t in $[0,T]$. Finally it remains only to note that since for each t $x(t)$ is independent of $B(u)-B(t)$ for $u \geq t$, (8.8), (8.10) and (8.13) together show that

$$E\left[e^{\lambda\int_t^{t+\Delta} x^2(u)du}\right] < \infty \quad \text{for} \quad 0 \leq t < t+\Delta \leq T.$$

The proof of the lemma is concluded by taking $\lambda = 16$.

The result just proved can be extended to system processes which are solutions of the stochastic equation

(8.14)
$$x(t) = x(0) + \int_0^t m[s,x(s)]ds + \int_0^t \sigma[s,x(s)]dB(s) ,$$

where in addition to (8.3) the following conditions are assumed.

(8.15) $m(t,\xi),\ \sigma(t,\xi)\quad (0 \le t \le T,\ -\infty < \xi < \infty)$ are continuous functions of (t,ξ);

$$0 < \sigma(t,\xi) \le K(1+\xi^2), \tag{8.16}$$

$$|\sigma(t,\xi_2) - \sigma(t,\xi_1)| \le K|\xi_1-\xi_2|$$

uniformly in t and

$$\int_0^a \frac{d\xi}{\sigma(t,\xi)} < \infty \tag{8.17}$$

for every $a > 0$ (with an analogous assumption if $a < 0$). Equation (8.14) can then be reduced to the form (8.2) considered in Lemma 8.1, viz,

$$y(t) = y(0) + \int_0^t m_0[s,y(s)]ds + B_0(t) ,$$

by means of the transformations

$$R(t,a) = \int_0^a \frac{d\xi}{\sigma(t,\xi)} \quad (a > 0),\quad y(t) = R(t,x(t))$$

and

$$m_0(t,y) = -\frac{r_t'(t,y)}{\sigma[t,r(t,y)]} + \frac{m[t,r(t,y)]}{\sigma[t,r(t,y)]} - \sigma_r'[t,r(t,y)] ,$$

where $a = r(t,y)$ is the inverse function of $y = R(t,a)$. The symbols r_t', σ_r' refer to the partial derivative of r with respect to t and of σ with respect to r. It will further be assumed that $m_0(t,y)$ satisfies the Lipschitz condition

$$|m_0(t,y) - m_0(t,y')| \le K_0|y-y'| . \tag{8.18}$$

The above conditions have been taken from G. Maruyama's paper [15]. We refer the reader to it for a discussion of this point and for conditions on the function m which ensure that (8.18) holds for m_0. Finally, condition (8.4) of Lemma 8.1 is to be replaced by

$$E\left[e^{cR^2(0,x(0))}\right] < \infty \qquad \text{for some } c > 0 . \tag{8.19}$$

It is then clear that the conclusion of Lemma 8.1 is valid for the process $x(t)$ which satisfies the stochastic equation (8.14) under the conditions (8.3), (8.15), (8.16), (8.17), (8.18) and (8.19).

Lemma 8.1 is no longer true without condition (8.4). In fact, (8.4) is necessary in order that a Markov process $x(t)$ which is a solution of (8.1) satisfy condition (6.5) of Theorem 6.1. More precisely, the following somewhat stronger assertion is true.

Lemma 8.4 Let $x(t)$ $(0 \le t \le T)$ be the solution of the stochastic differential equation (8.1) where the function m is assumed to satisfy (8.3). Further suppose that condition (6.5) of Theorem 6.1 holds for the process $x(t)$. Then, there exists a positive constant c such that

$$E\left[e^{cx^2(t)}\right] < \infty \tag{8.20}$$

for all t such that $0 \le t < t+\Delta \le T$.

Proof: From the relation

$$x(u) = x(t) + \int_t^u m[s,x(s)]ds + B(u)-B(t), \qquad (0 \le t < u)$$

we have

$$x^2(t) \le 2x^2(u) + 4\left(\int_s^u m[s,x(s)]ds\right)^2 + 4[B(u)-B(t)]^2;$$

from which, upon integrating both sides with respect to u, we have

$$hx^2(t) \le 2\int_t^{t+h} x^2(u)du + 4\int_t^{t+h} \left(\int_t^u m[s,x(s)]ds\right)^2 du + 4\int_t^{t+h} [B(u)-B(t)]^2 du.$$

From the Schwartz inequality and (8.3) it is easy to verify that

$$\int_t^{t+h} \left(\int_s^u m[s,x(s)]ds\right)^2 du = \int_0^h \left(\int_t^{t+u} m[s,x(s)]ds\right)^2 du$$

$$\le \int_0^h \left\{K^2u^2 + K^2u \int_t^{t+u} x^2(s)ds\right\}du$$

$$\le \frac{K^2h^3}{3} + \frac{K^2h^2}{2}\int_t^{t+h} x^2(s)ds \ ,$$

the last step following from the fact that $\int_t^{t+u} x^2(s)ds$ is an increasing function of u. Hence taking λ to be any positive number we have

$$E\left[e^{\lambda h x^2(t)}\right] \le e^{\frac{4K^2h^3\lambda}{3}} \; E\left[e^{2\lambda(1+\frac{K^2h^2}{2})\int_t^{t+h} x^2(u)du}\right]$$

$$\cdot \; E\left[e^{4\lambda\int_t^{t+h}(B(u)-B(t))^2du}\right].$$

By (6.5) the second factor on the right hand side of the above inequality is finite for all t $(0 \le t < t+\Delta \le T)$ if we take $h = \Delta$ and $\lambda = 8(1+K^2\Delta^2)^{-1}$. It has already been shown earlier (Lemma 8.3) that for $\lambda \le$ some λ_1 the last factor is finite for all t such that $0 \le t < t+\Delta \le T$, if $h = \Delta$. Thus the assertion is proved by setting $c = \Delta \cdot \min(\lambda_1, 8(1+K^2\Delta^2)^{-1})$.

We next turn to the identification of functions in $\mathcal{D}^*$ and the extended operator G_t^* (see Section 5) for the diffusion process $x(t)$ of Lemma 8.1.

Lemma 8.5 Let $x(t)$ satisfying

(8.21) $$\int_0^T E[x(t)]^2 dt < \infty$$

be the solution of the stochastic equation (8.14) where the functions m and σ satisfy conditions (8.3) and (8.15) - (8.19). If g, g' and g'' are bounded and continuous functions then

(8.22) $$g \in \mathcal{D}^*$$

and we may identify $G_t^* g$ as

(8.23) $$(G_t^* g)(x) = m(t,x)g'(x) + \tfrac{1}{2}\sigma^2(t,x)g''(x) .$$

Proof: The lemma will be proved if we show that for a g with bounded, continuous first and second derivatives, conditions (5.25), (5.26) and (5.27) are satisfied with G_t^* as in (8.23). First observe that Ito [7] has shown that for every x

$$\lim_{h \downarrow 0} \frac{(P_t^{t+h} g)(x) - g(x)}{h} = m(t,x)g'(x) + \tfrac{1}{2}\sigma^2(t,x)g''(x) ,$$

so that G_t^* defined by (8.23) is an "extension" of the weak or strong generalized infinitesimal generator of $x(t)$. Condition (5.25) and the Borel measurability in (t,x) of $(G_t^* g)(x)$ are both obviously satisfied, and (5.20) follows from (8.3), (8.15) and (8.21) since g' and g'' are bounded. It remains only to verify (5.27). It is known from the theory of stochastic differential equations (see e.g. [6], p. 514) that if the process $\xi_{s,x}(t)$ defined for $t \in [s,T]$ $(s \geq 0)$ is the solution of the equation

$$(8.24) \qquad \xi_{s,x}(t) = x + \int_s^t m[u,\xi_{s,x}(u)]du + \int_s^t \sigma[u,\xi_{s,x}(u)]dB(u),$$

then the $x(t)$ process which is the solution of (8.14) is a Markov process whose transition probability functions satisfy

$$P(s,x;t,A) = P[\xi_{s,x}(t) \in A].$$

Hence

$$(8.25) \qquad (P_s^t g)(x) - g(x) = \int g(y)P(s,x;t,dy) - g(x) = E\{g[\xi_{s,x}(t)]-g(x)\}.$$

We now apply the Ito lemma on stochastic differentials (Lemma 3.7) to the process $g[\xi_{s,x}(t)]$ where t lies in $[s,T]$. Recalling that g satisfies the assumptions of Lemma 3.7 and that $\xi_{s,x}(t)$ has the differential given by (8.24) we obtain

$$g[\xi_{s,x}(t)]-g(x) = \int_s^t \{g'[\xi_{s,x}(u)]m[u,\xi_{s,x}(u)]+\tfrac{1}{2}g''[\xi_{s,x}(u)]\sigma^2[u,\xi_{s,x}(u)]\}du$$

$$+ \int_s^t g'[\xi_{s,x}(u)]\sigma[u,\xi_{s,x}(u)]dB(u) .$$

Hence

$$(8.26) \qquad E\{g[\xi_{s,x}(t)] - g(x)\} = \int_s^t E\{g'[\xi_{s,x}(u)]m[u,\xi_{s,x}(u)]+\tfrac{1}{2}g''[\xi_{s,x}(u)]\sigma^2[u,\xi_{s,x}(u)]\}du .$$

The integral on the right hand side equals

$$\int_{-\infty}^{\infty} \{g'(y)m(u,y)+\tfrac{1}{2}g''(y)\sigma^2(u,y)\}P(s,x;u,dy)$$

$$= \int_{-\infty}^{\infty} (G_u^* g)(y)P(s,x;u,dy) \quad = \quad P_s^u G_u^* g)(x) \ .$$

The right hand side of (8.26) is thus equal to

$$\int_s^t (P_s^u G_u^* g)(x)du \ .$$

Condition (5.27) then follows from (8.25) and (8.26).

Lemma 8.6 Let $x(t)$ be the process of Lemma 8.5 with (8.21) replaced by

$$\int_0^T E[x(t)]^{8n+24}dt \ < \ \infty \ , \tag{8.27}$$

and such that

$$E[x(T)]^{8n+8} \ < \ \infty \ , \tag{8.28}$$

where n is a non-negative integer. Then conditions (7.11), (7.12), (7.13) and (7.14) of Theorem 7.1 are satisfied for $f(x) = x^n \cdot p(x)$, where p, p', p'' are bounded and continuous functions. Further, G_t^* applied to the functions x, $f(x)$ and $xf(x)$ is given by formally substituting $g(x) = x$, $f(x)$ and $xf(x)$ respectively in (8.23).

Proof: Clearly $p \in \mathcal{D}^*$ by the preceding lemma. To show that the (possibly unbounded) functions x, $f(x)$ and $xf(x)$ belong to $\mathcal{D}^*$ we proceed as in the proof of Lemma 8.5 making only some supplementary comments. First consider $g(x) = x$. Then

$$(P_s^t g)(x)-g(x) \quad = \quad E\{\xi_{s,x}(t)-x\} \quad = \quad \int_s^t E\{m[u,\xi_{s,x}(u)]\}du \ , \tag{8.29}$$

from (8.24). Setting

$$(G_u^* g)(x) \quad = \quad m(u,x) \ , \tag{8.30}$$

we have

$$E\{m[u,\xi_{s,x}(u)]\} = (P_s^u G_u^* g)(x)$$

and hence (5.27) follows from (8.29). The conditions (5.25) and (5.26) are obviously fulfilled from the assumptions on $x(t)$. Hence $x \in \mathcal{D}^*$. Since $f(x) = x^n p(x)$ and $xf(x) = x^{n+1} \cdot p(x)$ it suffices to show that the latter function is in $\mathcal{D}^*$. Writing $\varphi(x) = x^{n+1} p(x)$ for convenience we have

$$(8.31) \qquad (P_s^t \varphi)(x) - \varphi(x) = E\{\varphi(\xi_{s,x}(t)) - \varphi(x)\} .$$

Since $\xi_{s,x}(t)$ satisfies equation (8.24) for $t \in [s,T]$, (x fixed) we may apply Lemma 3.7 to the process $\varphi(\xi_{s,x}(t))$ to get

$$\begin{aligned} &\varphi(\xi_{s,x}(t)) - \varphi(x) \\ &= \int_s^t \{m[u,\xi_{s,x}(u)]\varphi'(\xi_{s,x}(u)) + \tfrac{1}{2}\sigma^2(u,\xi_{s,x}(u))\varphi''(\xi_{s,x}(u))\}du \\ (8.32) \qquad &+ \int_s^t \varphi''(\xi_{s,x}(u))\sigma[u,\xi_{s,x}(u)]dB(u) \\ &= \int_s^t a_{s,x}(u)du + \int_s^t b_{s,x}(u)dB(u), \text{ say .} \end{aligned}$$

From (8.27) it follows that the integrals $\int_s^t E|a_{s,x}(u)|du$ and $\int_s^t E[b_{s,x}^2(u)]du$ are finite. Taking expectations on both sides of (8.32) we obtain from (8.31) the relation

$$(8.33) \qquad (P_s^t \varphi)(x(s)) - \varphi(x(s)) = \int_s^t (P_s^u G_u^* \varphi)(x(s))du \qquad \text{a.s. } P_X ,$$

where

$$(8.34) \qquad (G_u^* \varphi)(x) = m(u,x)\varphi'(x) + \tfrac{1}{2}\sigma^2(u,x)\varphi''(x) .$$

The conditions (5.25) and (5.26) are verified easily and thus $\varphi(x) = xf(x) \in \mathcal{D}^*$. Hence condition (7.11) of Theorem 7.1 is satisfied. Conditions (7.12) and (7.13) follow immediately from (8.28), (8.27) and the boundedness of p. To prove (7.14) observe from (8.30) and (8.34) that

$$\int_0^T E[(G_t^* x)(x(t))]^8 dt = \int_0^T E[m(t,x(t))]^8 dt, \quad \text{and}$$

$$\int_0^T E[(G_t^* \psi)(x(t))]^8 dt = \int_0^T E\{m(t,x(t))\psi'(x(t)) + \tfrac{1}{2}\sigma^2(t,x(t))\psi''(x(t))\}^8 dt$$

where $\psi(x) = f(x)$ or $xf(x)$. Conditions (7.14) then follow from (8.27) and the assumptions (8.3) and (8.16) on m and σ and standard arguments involving the Schwarz and Hölder inequalities. This completes the proof of the lemma.

It is perhaps worth pointing out that we have not shown that

$$\text{(8.35)} \qquad \lim_{h \downarrow 0} \frac{(P_t^{t+h} g)(x) - g(x)}{h} = (G_t^* g)(x)$$

for $g(x) = x$, $f(x)$ or $xf(x)$. It can be easily seen that (8.35) holds under conditions on m and σ which ensure that the functions

$$(P_s^u m(u,\cdot)\varphi')(x) \quad \text{and} \quad (P_s^u \sigma^2(u,\cdot)\varphi''(x)$$

are (for fixed x) continuous in u $(s \le u \le t)$.

From the discussion in ([2] pp. 277-282) and Theorem 2 of [6] (p. 514) it can be shown that the process $x(t)$ satisfying the stochastic equation (8.14) and satisfying (8.3), (8.15) and (8.16) satisfies the measurability conditions (6.1) and (6.2) required in Theorems 6.1 and 7.1.

Piecing together the results proved in this section we are now in a position to show that our main theorems--Theorems 6.1 and 7.1--are applicable to a wide class of diffusion system processes. From Lemmas 8.1, 8.5 and Lemma 8.6 with $n = 0$ we obtain as an immediate consequence

<u>Theorem 8.1 Let $x(t)$ $(0 \le t \le T)$ be the diffusion process which is the solution of the stochastic equation (8.14). Further suppose assumptions (8.3), (8.15) - (8.19) are satisfied and that</u>

(8.36) $$\int_0^T E[x(t)]^{24} dt < \infty .$$

Then for every real function f which is bounded and continuous and has bounded and continuous first and second derivatives, the stochastic process $E^t[f(x(t))]$ *satisfies the Ito stochastic differential equation* (6.74) *of Theorem* 6.1 *and the Fisk-Stratonovich equation* (7.15) *of Theorem* 7.1.

The second consequence of Lemmas 8.1, 8.5 and 8.6 (with $n \geq 1$) is the following result which yields stochastic equations for $E^t[x^n(t)g(x(t))]$.

Theorem 8.2 *Let* $x(t)$ *be the diffusion process of the preceding theorem where again conditions* (8.3), (8.15) - (8.19) *are assumed to hold. Further suppose that* $x(t)$ *satisfies* (8.27). *If* g, g' *and* g'' *are bounded and continuous,* $E^t[x^n(t)g(x(t))]$ $(n \geq 1)$ *satisfies the following Ito stochastic differential equation*

$$\begin{aligned} dE^t[x^n(t)g(x(t))] &= E^t[(G_t^* x^n g)(x(t))]dt \\ &\quad + \{E^t[x^{n+1}(t)g(x(t))] - E^t[x^n(t)g(x(t))]E^t[x(t)]\} \\ &\quad \cdot \{dz(t) - E^t[x(t)dt\}, \qquad 0 \leq t \leq T . \end{aligned} \tag{8.37}$$

The corresponding stochastic equation of the Fisk-Stratonovich type (7.15) *with* $f = x^n g$ *also holds. In particular, if* $n = 1$ *and* $g = 1$ *we have the Ito equation*

$$\begin{aligned} dE^t[x(t)] &= E^t[m(t,x(t))]dt \\ &\quad + \{E^t[x^2(t)] - (E^t[x(t)])^2\}(dz(t) - E^t[x(t)]dt) \quad 0 \leq t \leq T . \end{aligned} \tag{8.38}$$

Remark. It is interesting to seek conditions on the coefficients m and σ of the diffusion equation (8.14) which would ensure the validity of the moment conditions (8.27) and (8.36). In many practical situations it is natural to assume that $\sigma[t,x(t)]$ does not depend on $x(t)$ or more generally that $\sigma[t,x(t)]$ is bounded with respect to x.

For diffusion system processes of this class the following lemma shows that moments of all order of $x(t)$ exist and are integrable with respect to t over $[0,T]$.

Lemma 8.7 *Let* $x(t)$ *be the diffusion process satisfying* (8.14) *where conditions* (8.3), (8.15), (8.17), (8.18), (8.19) *are assumed to hold and* (8.16) *is replaced by*

$$(8.16)' \qquad \begin{aligned} &0 < \sigma(t,\xi) \le K < \infty, \\ &|\sigma(t,\xi_1) - \sigma(t,\xi_2)| \le K|\xi_1-\xi_2|. \end{aligned}$$

for all t *and* ξ. *Then for every integer* $n \ge 1$,

$$(8.39) \qquad E|x(t)|^n < \infty$$

for all t, *and*

$$(8.40) \qquad \int_0^T E|x(t)|^n dt < \infty .$$

Proof: From (8.16)', (8.19) and recalling that

$$R(t,x(t)) = \int_0^{x(t)} \frac{d\xi}{\sigma(t,\xi)}$$

it follows that $E\{e^{cx^2(0)}\} < \infty$ for some $c > 0$. From the remarks made at the end of Lemma 8.1 it suffices to prove the lemma for $x(t)$ satisfying equation (8.2) instead of (8.14). Lemmas 8.1 and 8.4 then together imply that for some $\lambda > 0$ and all t in $[0,T]$ $E[e^{\lambda x^2(t)}] < \infty$. This yields (8.39). To prove (8.40) we proceed as in the proof of Lemma 8.1. For t in $[0,T]$ and $t \le u \le T$, and without loss of generality assuming n to be an even integer we have from (8.2)

$$(8.41) \qquad \begin{aligned} |x(u)-x(t)|^n &\le K_1 \cdot |u-t|^{n-1} \int_t^u (1+|x(s)|^n)ds \\ &+ 2^{n-1}|B(u)-B(t)|^n \qquad (K_1 = K^n \cdot 2^{\frac{3n}{2}-2}). \end{aligned}$$

Hence, choosing $h > 0$ such that $K_2 = 1 - \frac{2^{n-1}K_1h^n}{n} > 0$ we obtain from (8.41)

$$(8.42)\qquad K_2 \int_t^{t+h} |x(u)|^n du \le \frac{2^{n-1}K_1h^{n+1}}{n} + 2^{2n-2}\int_t^{t+h} |B(u)-B(t)|^n du + 2^{n-1}h|x(t)|^n .$$

From (8.39) and since clearly

$$\int_t^{t+h} E|B(u)-B(t)|^n du < \infty$$

it follows upon taking expectations of both sides of (8.42) that

$$(8.43)\qquad \int_t^{t+h} E|x(u)|^n du < \infty .$$

Letting $N = [\frac{T}{h}]$ (the largest integer in T/h) and substituting $t = 0, h, \ldots, (N-1)h$ in (8.43), we deduce the finiteness of

$$\int_0^{Nh} E|x(u)|^n du .$$

Also from (8.43)

$$\int_{Nh}^{T} E|x(u)|^n du < \infty.$$

Hence (8.40) follows and the lemma is proved.

Theorem 8.2 is of statistical interest, particularly when the system process $x(t)$ is Gaussian. In this case $E^t[x(t)]$ which is the conditional mean of $x(t)$ given the data up to time t and which we shall denote by $\hat{x}(t)$ is the best estimate (with quadratic loss function) of the state $x(t)$ of the system. We thus obtain as a corollary from Theorem 8.2 the following result due to Kalman and Bucy [10].

<u>Corollary</u> <u>Let $x(t)$ be the (Gaussian) process given by the equation</u>

$$(8.44)\qquad dx(t) = m(t)x(t)dt + \sigma(t)dB(t), \qquad (0 \le t \le T)$$

<u>where m, σ and σ' are non-random continuous functions of t, and</u>

σ is strictly positive. Then $\hat{x}(t)$ satisfies the Ito stochastic differential equation

$$dx̂(t) = m(t)\hat{x}(t)dt + E^t[x(t)-\hat{x}(t)]^2(dx(t)-\hat{x}(t)dt) \quad 0 \le t \le T. \tag{8.45}$$

It should be noticed that the corollary does not give the entire Kalman-Bucy result. It remains to be shown that the conditional distribution of $x(t)$ given $\mathcal{G}_z^t$ is Gaussian. Once this is done, the ordinary differential equation for the conditional covariance (variance in our one-dimensional case) follows easily from Theorem 8.2. Since showing that the conditional distribution is Gaussian involves the more general question of the unique determination of the conditional distribution by the Ito equation (6.74) of Theorem 6.1, discussion of this point will be delayed to a later paper.

The property (6.5) remains the most stringent condition we have had to impose on the system process $x(t)$ and the reader will observe that crucial use has been made of it in the proofs of our basic theorems. It clearly holds if $x(t)$ is a Markov process whose state space may be discrete or continuous but which is uniformly bounded a.s. with respect to t. Thus the results of the two cases treated separately by Wonham and Shiryaev can both be deduced from Theorem 6.1. Another feature of condition (6.5) is that if $x_1(t)$ and $x_2(t)$ are two processes satisfying the condition, their sum $x(t) = x_1(t)+x_2(t)$ also satisfies it. An example of a process for which the condition does not hold is the homogeneous Poisson process. Since almost all sample functions of the Poisson process are non-decreasing functions, for every positive Δ,

$$E\left[e^{16\int_t^{t+\Delta} x^2(u)du}\right] \ge E\left[e^{16\Delta x^2(t)}\right].$$

However, it can be easily verified that the right hand side expectation is infinite for every $t > 0$. It is therefore desirable to relax this restriction so as to make Theorems 6.1 and 7.1 applicable to system processes including the Poisson and other discrete-valued Markov processes. It can be shown that (6.5) is not necessary by constructing a process to which Theorem 6.1 applies but which does not satisfy (6.5)

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